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HOLOGRAPHIC CONSIDERATIONS ON RIEMANN-CARTAN SPACETIME

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Abstract

Theories of gravity and matter fields on asymptotically-AdS spacetimes have received considerable attention due to the famous AdS/CFT correspondence. Traditionally, the majority of the related work assumes that the geometry of the bulk spacetime is Riemannian (i.e., torsion-free), and gravity is described using the second-order formalism, where the fundamental field is the metric $g_{\mu\nu}$. However, there are situations where the Riemann-Cartan spacetime and the first-order formulation, with the vielbein and the spin-connection as fundamental variables, are more useful, if not necessary, for addressing issues in formulating the gravity theory. The most common setup where the first-order formulation becomes important is the consideration of fermions in a curved background. However, if we are willing to go beyond the standard Einstein-Hilbert gravity, torsion may not vanish even in a vacuum. Furthermore, one way to define a quantum spacetime is to introduce noncommutative (NC) geometry, and it is often useful to use the vielbein and the spin-connection to define a model of NC gravity.

The main goal of this PhD thesis is to study gravity and matter fields on (asymptotically) AdS spacetimes, possibly with torsion, in the first-order formalism. More concretely, we will explore Chern-Simons gravity and the closely related model of Chamseddine's even-dimensional topological gravity, including black hole solutions, boundary one-point functions, and branes. We will spend considerable time discussing boundary terms in the gravitational action. Finally, we will discuss conductivity in the boundary theory, and we shall also see how the methodology used in this thesis can be applied to study an NC AdS spacetime as a bulk in some sort of an AdS/CFT duality.

Key words: *Riemann-Cartan spacetime, AdS/CFT correspondence, first-order gravity formalism, Chern-Simons gravity, noncommutative gravity*

Research field: **physics**

Research subfield: **high energy physics**

Apstrakt

Teorije koje analiziraju gravitaciju i polja materije na asimptotski-AdS prostorima su intenzivno izučavane zbog čuvene AdS/CFT korespodencije. Tradicionalno, većina radova u ovoj oblasti podrazumeva da je geometrija prostor-vremena Rimanova i da je gravitacija formulisana koristeći formalizam drugog reda, gde je fundamentalno polje metrika $g_{\mu\nu}$. Međutim, postoje slučajevi gde je formalizam prvog reda, sa filbajnom i spin-koneksijom kao fundamentalnim poljima, korisniji, ako ne i neophodan, za formulisanje teorije gravitacije. Najpoznatiji primer ovakve situacije je razmatranje fermiona na zakrivljenom prostor-vremenu. Međutim, ako smo voljni da analiziramo teorije opštije od Ajnštajn-Hilbertove gravitacije, torzija može postojati čak i bez uvođenja dodatnih polja materije. Pored toga, jedan način da se formuliše kvantno prostor-vreme je formalizam nekomutativne geometrije i ispostavlja se da je često upravo formalizam prvog reda koristan da bi se ovakav model napravio.

Stoga, glavni zadatak ove doktorske teze je proučavanje gravitacije i drugih polja na (asimptotski) AdS prostor-vremenu, često sa torzijom, koristeći formalizam prvog reda. Konkretnije, istraživaćemo Čern-Sajmons gravitaciju i usko povezan model Šamsedinove topološke gravitacije, crne rupe u tim modelima, jednotačkaste funkcije na granici i brane. Provešćemo dosta vremena diskutujući granične članove u gravitacionim teorijama. Na kraju, diskutovaćemo provodnost u dualnoj graničnoj teoriji i razmatraćemo kako se metodologija razvijena u ovoj tezi može iskoristiti za bolje razumevanje nekomutativnog AdS prostora u kontekstu holografske dualnosti.

Ključne reči: *Riman-Kartanovo prostor-vreme, AdS/CFT korespodencija, gravitacija u formalizmu prvog reda, Čern-Sajmons gravitacija, nekomutativna gravitacija*

Naučna oblast: **fizika**

Uža naučna oblast: **fizika visokih energija**

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As English is not my first language, I made use of the software Grammarly to check for spelling errors and improve the clarity of the presentation; however, no new sentences were generated using AI assistance in this thesis, and I take the full responsibility for all the comments made herein.

To my wife, Ivana

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Chapter 1

Introduction

There is a tale, usually told to undergraduate students, that one can ignore all the boundary terms arising in some variational calculations as they are deemed irrelevant. Of course, no one would, having faced the question of whether the previous statement is true, support this claim in full validity, yet many student textbooks do not spend any time discussing the nature of the boundary terms that arise, for example, from a variation of an action [1]. However, for many purposes, boundary terms play an important role. One example is the application of the second Noether's theorem in gauge theory to obtain conserved charges [2]. Variation of an action generically contains a bulk term, which is zero on-shell (this is precisely how we derive equations of motion), and a boundary term, which gives something called *the pre-symplectic potential*. We can illustrate the usage of boundary terms in a more elementary example of classical mechanics. If we consider action as a function of coordinates and time, we are interested in variations of the action at the boundary, and it is precisely this quantity that defines momentum in this approach. To illustrate this, note that in the mechanics of a single particle, we have

$$\delta\left(\int_{t_i}^{t_f} L(q,\dot{q},t)\mathrm{d}t\right) = \int_{t_i}^{t_f}\left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\right)\delta q + \left[\frac{\partial L}{\partial \dot{q}}\right](t_f)\delta q(t_f) - \left[\frac{\partial L}{\partial \dot{q}}\right](t_i)\delta q(t_i)\,, \quad (1.1)$$

where the integral vanishes on-shell, and the boundary contribution precisely gives a well-known expression for the canonical momentum $p = \frac{\partial L}{\partial \dot{q}}$. The (most probable) reason why we don't always care about boundary terms is that we implicitly assume that appropriate boundary conditions are imposed such that those boundary terms, produced from the variation of the action, vanish. Indeed, boundary terms are closely related to boundary conditions. On the other hand, the role of boundary conditions is often stressed, both in the mathematical and physical literature. It is, therefore, evident that, despite our initial claim, the boundary terms are very important.

One particular area of research in which boundary terms and boundary conditions are of great importance is gravity. Traditionally formulated as a classical theory of spacetime geometry, this interesting theory is, assuming a specific set of asymptotic boundary conditions, conjectured to be related to the quantum field theory (QFT) living at the boundary of a spacetime [3]. There are many hints that this is indeed true, and we shall spend some time in this thesis elaborating along those lines. Because of this relation between a bulk theory with gravity and the corresponding boundary QFT, one has to devote considerable attention to the study of boundary terms and boundary conditions in gravity. Even without this duality in mind, conserved quantities in gravity, which determine black holes' mass, angular momentum, and entropy, are given by boundary integrals, thus highlighting the importance of boundary terms in gravity [2]. In this thesis, we will study gravity on Riemann-Cartan spacetimes (including torsion) in the first-order formalism. Because of the mentioned duality, we will be interested only in a certain type of spacetimes, those called *asymptotically AdS* spacetimes, and deformations thereof. The study conducted as a part of this PhD thesis, though being original, relies on a majority of the related work done in the past decades, making it obligatory to go through some of the most important results that predate our research.

This thesis is organized as follows. In Chapter 1, we go through the historic route of major scientific developments leading to the work presented here, and focus on motivating the topic of this PhD thesis thoroughly. Chapter 2 establishes the methodology for the study of Riemann-Cartan spacetime that is used throughout the thesis. Chapter 3 introduces the bottom-up AdS/CFT and contains important examples that set up the stage for later original calculations. In Chapter 4, we study Chamseddine's even-dimensional topological gravity from a holographic point of view and its two-dimensional example - JT gravity. In Chapter 5, we analyse Chern-Simons gravity with torsion on asymptotically AdS spacetime, AdS/BCFT, reduction to two dimensions, and black hole entropy. In Chapter 6, we propose a model of a nonminimally coupled $U(1)$ gauge field that is sensitive to torsion and compute the holographic conductivity in a dual field theory. Finally, in Chapter 7, we analyse noncommutative gravity, the role of torsion in the formulation of some versions of this theory, and boundary correlation functions in the case when the AdS bulk is given as a quantum spacetime, with emphasis on those cases where the noncommutativity is introduced using a framework based on the first-order formulation of gravity. Final remarks and proposals for further research are discussed in Chapter 8.

1.1 Historical overview

In order to understand our research goal, we shall go through a highly biased historical timeline leading to this thesis. Some important points are depicted in Figure 1.1. It is often stated that classically, the study of gravity is actually the study of geometry [4]. This study was initiated by Albert Einstein's general relativity, where the equations of motion determining the geometry from the matter content are the so-called Einstein's equations

$$R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu} \,, \tag{1.2}$$

where $T^{\mu\nu}$ is the matter energy-momentum tensor, and $R^{\mu\nu}$ and R are, respectively, the Ricci tensor and the Ricci scalar for the metric $g_{\mu\nu}$. After writing down the equations of motion for a gravitational field, Einstein understood that there is a problem with finding a static solution that should explain our universe, unless an additional constant is introduced in the theory [5]. With that constant, denoted by Λ, equations of general relativity take the form

$$R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \,. \tag{1.3}$$

However, it was soon realized that our universe is not static, and therefore, there is no need to include the cosmological constant in the theory.

Some of the most interesting solutions to vacuum equations (1.2) are black holes (BH). Even though it is challenging to give a completely general definition of a BH, it is fairly simple to recognize one in practice. Black holes are characterized by the existence of an event horizon and

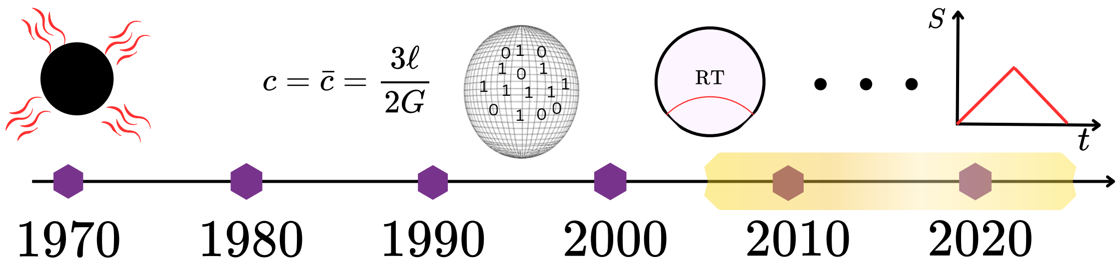

Figure 1.1: A chronological order of subjectively chosen events relevant for this work. Holography on Riemann-Cartan spacetime has been analyzed only in the highlighted period of time.

a singularity surrounded by this horizon. A prototype of such a spacetime is the Schwarzschild black hole, with the line element given by

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{1.4}$$

where $f(r) = 1 - \frac{2MG_N}{r}$ (we will mostly set $G_N = 1$ in the following). The position of the horizon is determined by $f(r_h) = 0$, which for the case of the Schwarzschild BH gives $r_h = 2M$. More general BH spacetimes we will consider in this thesis have a similar form of the line element. Interestingly, it turns out that BHs behave as thermodynamic systems, which was discovered in the early 1970s. To illustrate this, we start by noticing that the horizon is smooth in the Lorentzian signature and that there is no singularity (except for the coordinate one) at this location. We, therefore, expect that a similar conclusion should hold in the Euclidean signature, once the Wick rotation $t \to -i\tau$ is performed. In Euclidean signature, we have

$$\mathrm{d}s_E^2 = f(r)\mathrm{d}\tau^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{1.5}$$

Near the horizon $r = r_h$ we have $f(r) \approx 0 + (r - r_h)f'(r_h) + \dots$. We introduce the radial coordinate

$$\rho = \int_{r_h}^{r} \frac{\mathrm{d}r}{\sqrt{f(r)}} \approx \frac{2}{\sqrt{f'(r_h)}}\sqrt{r - r_h}\,, \tag{1.6}$$

such that $f(r) \approx \left(\frac{1}{2}f'(r_h)\right)^2 \rho^2$. Expression $\kappa \equiv \frac{1}{2}f'(r_h)$ is called the *surface gravity*, and plays an important role in the BH's thermodynamics. Euclidean line element now takes the form

$$\mathrm{d}s_E^2 \approx \rho^2\kappa^2\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + r_h^2\mathrm{d}\Omega^2\,. \tag{1.7}$$

The τ-ρ part of the metric corresponds to a plane in polar coordinates. In order to avoid conical singularities, the circumference of a circle in τ coordinate must be $2\pi\rho$. However, from (1.7), it follows that this circumference is $\rho\kappa\beta$, where β is the period of the τ coordinate. This implies that

$$\beta = \frac{2\pi}{\kappa}\,. \tag{1.8}$$

On the other hand, it is well-known from the study of finite temperature QFT that the period of the Euclidean time coordinate equals the inverse temperature, $\beta = \frac{1}{T}$. Therefore, the temperature of a BH is given by

$$T = \frac{\kappa}{2\pi} = \frac{f'(r_h)}{4\pi}\,. \tag{1.9}$$

In the case of the Schwarzschild black hole, we get $T = \frac{1}{8\pi M}$. Furthermore, by interpreting M as the BH mass, using Einstein's formula $E = M$ (in units where $c = 1$), and referring to the first law of thermodynamics $T\mathrm{d}\mathcal{S} = \mathrm{d}E$, we get

$$\mathrm{d}\mathcal{S} = 8\pi M \mathrm{d}M \ . \tag{1.10}$$

Assuming that the entropy of a BH vanishes for vanishing mass, we obtain

$$\mathcal{S} = 4\pi M^2 = \frac{A}{4} \ , \tag{1.11}$$

where $A = 4\pi r_h^2 = 16\pi M^2$ is the area of the horizon. Including all the constants we have stripped away, the result is

$$\mathcal{S} = \frac{k_b c^3}{4\hbar G} A \ . \tag{1.12}$$

This formula, connecting the BH entropy and the horizon's area, is called the Bekenstein-Hawking entropy (BH entropy for short). The previous derivation is well-known, presented in various books and lecture notes, appearing even in high-school physics competitions [6]. Historically, the first hint that the area A behaves as entropy was discovered in BH mechanics, relying on the work of Hawking and collaborators in the early 1970s [7]. At that time, it was believed that BHs, as classical objects, had zero entropy. The theoretical discovery of BH evaporation (Hawking radiation [8]) demonstrated that the notion of BH temperature indeed has a physical interpretation as a real temperature. Further progress in this direction was made when Gibbons and Hawking derived the BH's entropy from the gravitational path integral in the saddle point approximation [9]. This approximation means that we consider a path integral over a metric field and evaluate it as

$$\int \mathcal{D}g_{\mu\nu} e^{-S} \approx e^{-S_{\mathrm{on-shell}}} \ , \tag{1.13}$$

where we don't care about the exact formulation of the path integral and the measure $\mathcal{D}g_{\mu\nu}$. An important part of this construction is played by the Gibbons-Hawking-York (GHY) term, which is a boundary term introduced in [9, 10]. While the primary role of this boundary term is to implement Dirichlet boundary conditions on the metric field, on a technical level here, it makes $S_{\mathrm{on-shell}}$ non-zero. Magically, this construction yields the correct result for the BH entropy starting from a single classical state (spacetime), contrary to the standard thermodynamic interpretation of entropy, in which large entropy is associated with a large number of microstates. This problem persists to this day, and a huge bulk of modern research is dedicated to understanding the BH microstates.

Despite its gloomy origin, formula (1.12) provides an essential insight into the quantum properties of gravity. Let us compare this formula with the Sackur-Tetrode formula [11] for the entropy of N molecules of an ideal gas in volume V, with concentration $n = \frac{N}{V}$ and energy per molecule ϵ:

$$\mathcal{S} = k_b N \left(\ln \left(n \left(\frac{m\epsilon}{3\pi\hbar^2} \right)^{\frac{3}{2}} \right) + \frac{5}{2} \right) \ . \tag{1.14}$$

In this case, entropy is proportional to the total number of molecules, uniformly distributed over the bulk of volume V. This is clearly different from (1.12), which is proportional to the area surrounding the BH region. One may even conclude from this formula that the gravitational degrees of freedom are not distributed over the volume, but rather over the boundary enclosing it. From the analogy of this statement with the electromagnetic holograms, which encode three-dimensional objects in two-dimensional pictures, we call the related idea about gravity the holographic principle. This idea was put forward by t'Hooft and Suskind in the early 1990s [12, 13]. Through this thesis, we will assume that gravity is holographic, even though we must stress that this is not accepted as fact in all physics communities. For example, the fact that area instead of volume appears in (1.12) is natural, knowing that there is not a single candidate for the diffeomorphism-invariant notion of spatial volume in general relativity, but there are works showing how the notion of volume may naturally be generalized to curved spacetime [14]. This volume can be very large and, in principle, the fact that the thermodynamical entropy is proportional to the area does not necessarily imply the holographic nature of gravity; degrees of freedom might, after all, be distributed over this large volume and their number can be considerably larger than previously believed.

Another important, although at first sight unrelated, observation concerning gravity and boundaries was carried out in the mid-1980s, by Brown and Theietelbohm [15]. They considered three-dimensional gravity with a negative cosmological constant $\Lambda < 0$. A maximally symmetric solution to Einstein's equation with negative cosmological constant is called the Anti de-Sitter (AdS) spacetime. By analyzing the algebra of asymptotic symmetries of this theory, they found out that it is given by two copies of the Virasoro algebra with central extensions $c = \overline{c} = \frac{3\ell}{2G_N}$ (ℓ being the AdS radius). Actually, in the case of four-dimensional asymptotically flat gravity, a similar analysis was carried out even before, by Bondi and collaborators [16, 17, 18].

A dramatic change of a viewpoint on the relation between boundaries and gravity happened in late 1997, with Maldacena's conjecture on the relation between String theory in $AdS_5 \times S^5$ background and large N four-dimensional $\mathcal{N} = 4$ super Yang-Mills: the AdS/CFT duality [3]. This duality establishes a connection between a theory of gravity (string theory in a concrete realization) and a quantum field theory, defined on the boundary of AdS spacetime, and thus living on a manifold with one less dimension. This proposal also provided a concrete realization of the holographic principle [19] and is the first attempt to state that the boundary charges are obtained from an "ordinary" QFT. Soon after [3], additional insights into the relation between boundary observables and bulk fields were made in [20, 21], resulting in the so-called GKPW (Gubser-Klebanov-Polyakov-Witten) dictionary of holographic duality. This dictionary establishes that the bulk partition function, obtained as an Euclidean path integral over the metric and all other fields, with appropriate AdS boundary conditions, equals the CFT partition function in the presence of sources. On the other hand, sources in the boundary theory are obtained from bulk fields (ϕ) at the boundary (ϕ_0), so the GPKW dictionary states that

$$Z_{AdS}(\phi \to \phi_0) = Z_{CFT}(\phi_0) = \left\langle e^{\int \mathrm{d}x^{D-1} \phi_o \mathcal{O}} \right\rangle_{CFT} , \tag{1.15}$$

with $\mathcal{O}$ being a boundary QFT operator dual to the bulk field ϕ. We will be more precise about the precise meaning of (1.15) in section 3. For example, if ϕ is a bulk scalar field with mass m, $\mathcal{O}$ is a scalar boudnary operator with conformal dimension $\Delta = \frac{1}{2}\left(d + \sqrt{4m^2/\ell^2 + d^2}\right)$. If ϕ is a $U(1)$ bulk gauge field, $\mathcal{O}$ is a boundary conserved current, providing the boundary theory with $U(1)$ global symmetry. Finally, if ϕ is a metric field, $\mathcal{O}$ is a stress-energy tensor.

Following these initial considerations, a plethora of papers appeared in the literature on AdS/CFT duality, and Maldacena's paper [3] is among the most cited in theoretical high-energy physics. Throughout the thesis, we will also refer to this type of duality as gauge/gravity duality. Based on the GPKW dictionary, it is evident that one can try to use the AdS/CFT in a setting that does not include String theory or supersymmetry - simply by using a saddle point approximation on the left-hand side of (1.15), analogous to (1.13). By relying on a (semi-)

classical bulk theory, we can use symmetry as a guiding principle to build a bulk model that yields the desired boundary behavior. The resulting *bottom-up* approach enabled researchers to build models of condensed-matter systems, resulting in the field of AdS/CMT (Anti-de Sitter/ condensed matter theory) correspondence. The primary example of this type of consideration is given by a holographic superconductor [22]. Furthermore, classical gravity and geometry can be used to compute a quantity in a boundary theory that is inherently quantum mechanical - the entanglement entropy, as conjectured by Ryu and Takayanagi in 2006 [23].

Of course, it is unfair to state that the only important development in our understanding of gravity is related to AdS/CFT duality. Ever since Einstein formulated it, scientists have sought a more advanced theory of gravity that would ultimately resolve the problems posed by Einstein's original formulation. Einstein himself formulated the teleparallel theory of gravity [24], where torsion, a geometric quantity initially set to zero in GR, plays an important role. Precisely, this object plays an important role in this thesis, as we combine the ideas of bottom-up AdS/CFT and torsion. Before discussing the motivation for our work, let us comment on the history of merging AdS/CFT with torsion. As far as we are aware, the first paper trying to analyze torsionful bulk in the context of AdS/CFT duality is [25], focusing on the five-dimensional Chern-Simons (CS) gravity. Following this work, another similar paper discussed three-dimensional gravity with torsion [26]. Another paper, technically distinct from the previous two, emerged the following year [27, 28]. In the following ten-year period, only a couple of papers on this topic appeared, such as [29, 30]. Recently, [31] considered the GHY terms in theories with torsion (and nonmetricity), with the idea of better understanding boundary terms needed for studying holography on such spacetimes. Finally, there is a recent interest in the Kerr/CFT correspondence with torsion [32]. This review sets up the stage for our PhD work, where, in addition to torsion, we will also include noncommutative deformations of spacetime, as we shall discuss later.

1.2 Motivation

As this thesis's main focus is holography (in the AdS/CFT sense) on Riemann-Cartan spacetime, we must thoroughly motivate our study. While most of the research directions we review in this section are well-studied, there is a limited number of works that attempt to combine them. Therefore, we will provide a comprehensive overview of the techniques used throughout this text. In this section, we start considering the main ingredients of our work: gauge theories of gravity, Chern-Simons theory, noncommutative (NC) gravity, and AdS spacetime.

1.2.1 Gauge theory of gravity

General relativity is invariant under diffeomorphisms. Diffeomorphisms are local transformations in the sense that the parameter does not have to be a spacetime constant. Despite this, it is not entirely correct to say that GR is a gauge theory of diffeomorphisms in the same sense that YM is a gauge theory for a certain Lie group [33]. First of all, diffeomorphisms do not form a Lie algebra. Furthermore, YM gauge theory is based on the localisation of global symmetries, while it is not a priori obvious if gravity can be obtained by localising some global symmetries. As the most well-studied global spacetime symmetry is the Poincaré symmetry, one can hope that gravity can be formulated by localising this symmetry. The answer is, in a certain sense, positive, and there is indeed a research direction titled Poincaré gauge theory of gravity [34, 35]. In essence, it introduces gravitational degrees of freedom through vielbein and spin-connection as independent fields (see Chapter 2 for the details). The spin-connection is obtained as a gauge field for local Lorentz transformations, while the vielbeins are related to translations. However, a specific way (on which we will not elaborate here) in which vielbein fails to be a proper gauge field renders Poincaré gauge theory not adaptable to the fiber bundle structure, as opposed to YM theory. Classically, this theory is similar to GR; the difference lies in the fact that the torsion tensor is not set to zero in general. This is obvious by construction as vielbeins and spin-connection are introduced independently. However, if one considers the Einstein-Hilbert (EH) action, the equations of motion actually imply that the on-shell torsion vanishes. An additional matter field (e.g., Dirac spinor) may imply that the torsion is non-vanishing, though it is entirely determined by the matter content. However, additional terms in the action can result in different conclusions, and torsion can end up being dynamical. A similar approach, where vielbeins and spin-connection are treated independently, though we do not assume that gravity is obtained from the localisation of the Poincaré group, is called the *first-order* formulation of gravity.

Therefore, Einstein-Hilbert gravity in four dimensions is not a *proper* gauge theory. The situation is different in three dimensions. In this case, EH gravity can be formulated as a proper gauge theory. This is a consequence of the fact that classical three-dimensional gravity can be rephrased as a Chern-Simons gauge theory [36]. We will explain the details of this construction later, but here we note that CS is a particular gauge theory (even though here we discuss the case of three spacetime dimensions, we will explain how to generalize CS gravity to a higher number of dimensions). The gauge group in this case is indeed a three-dimensional Poincaré group. This claim can be generalized to include the cosmological constant (both positive and negative), and the respective gauge group in this case is $SO(3,1)$ and $SO(2,2)$, respectively. In two dimensions, JT gravity, which is a particular case of two-dimensional dilaton gravity with cosmological constant, can also be formulated as a gauge theory [37]. In this case, the relevant gauge theory is BF theory in two dimensions, and the gauge group is $SO(2,1)$ or $SO(1,2)$. Additionally, there is an approach to four-dimensional EH gravity with a cosmological constant that uses a gauge-theoretic formulation with appropriate symmetry breaking, called the MacDowell-Mansouri-Chamseddine-Stelle-West approach [38, 39]. In all those situations, the gauge connection takes the form

$$A = \frac{1}{2}\omega^{ab} J_{ab} + \frac{1}{\ell} e^a P_a \ , \tag{1.16}$$

where J_{ab} are generators of Lorentz transformations and P_a are generators of (generalized, in the case of AdS or dS group) translations. Once this connection is used in the gauge theory action, the resulting theory should match the gravitational one. In the next subsection, we turn to the most important example that can be used to formulate gravity: the CS gauge theory.

1.2.2 Chern-Simons theory

Consider a gauge group G with generators T_a and a trivial G-bundle over a three-dimensional manifold $\mathcal{N}$. Let A be a Lie-algebra-valued one-form $A = A^a_\mu \mathrm{d}x^\mu T_a$. Let us focus on a gauge theory with the action given by

$$S = \frac{k}{4\pi} \int_{\mathcal{N}} \mathrm{Tr} \left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right) \ . \tag{1.17}$$

For concreteness, we start analysing $G = U(1)$. The action in this case comes down to

$$S = \frac{k}{4\pi} \int_{\mathcal{N}} A \wedge \mathrm{d}A \,. \tag{1.18}$$

Note that the Lagrangian form can be written as $A \wedge \mathrm{d}A = \mathrm{d}(F \wedge F)$, assuming we are writing the last equality on a four-dimensional manifold. Stationary points of this action give $F = \mathrm{d}A = 0$. Therefore, on-shell, the gauge connection is flat. This may seem to imply that this theory is too simple and irrelevant for the description of some physically interesting phenomena. However, CS theory plays an important role in many different aspects of physics and mathematics. For example, this theory is used in condensed matter physics when describing the quantum Hall effect [40]. Furthermore, as we previously explained, CS theory is related to three-dimensional gravity. Finally, this theory is related to knots and their invariants, which establishes a direct link between physics and mathematics [41].

Action (1.17) does not contain a metric tensor, and in this sense, we are dealing with a topological field theory. Whether this action is gauge-invariant is a nontrivial question, as it involves the gauge connection explicitly. Even though (1.18) is invariant under $\delta A = \mathrm{d}\lambda$ up to boundary terms, as

$$A \wedge \mathrm{d}A \to A \wedge \mathrm{d}A + \mathrm{d}(\lambda \wedge \mathrm{d}A) \,,$$

gauge invariance is not so obvious in the non-Abelian case. Interestingly, both Abelian and non-Abelian cases turn out to be nontrivial, and the issue of gauge invariance is solved at the quantum level only when the constant k, usually referred to as the CS *level*, is quantized to be an integer. This integer is related to the conductivity in the quantum Hall effect. The most important observables in this type of gauge theories are not described by local operators, but rather by extended ones: *Wilson loops and lines*. Wilson line describes a parallel transport of a heavy charged particle in the external gauge field A. In the case of a $U(1)$ gauge theory, they are defined as

$$W(x_a, \gamma, x_b) = e^{iq \int_{x_a}^{x_b} A_\mu \mathrm{d}x^\mu} = e^{iq \int_{x_a}^{x_b} A} \,. \tag{1.19}$$

Parameter q is the charge that labels an irreducible representation of the $U(1)$ group. When $x_a = x_b$ we get the Wilson loop $W(\gamma) = e^{iq \oint_\gamma A}$. In the case of a non-Abelian group, the definition for the Wilson loop is

$$W(\gamma) = \mathrm{Tr}_{\mathbf{R}} \mathcal{P} e^{iq \oint_\gamma A} \,, \tag{1.20}$$

with $\mathcal{P}$ standing for the path ordering. Wilson loops depend on the choice of the representation $\mathbf{R}$ of the gauge group. It is often stated that Wilson loops in CS theory are topological. Let us illustrate this claim. Consider two Wilson lines between points x_a and x_b, oriented in opposite directions. We label one path as γ_1 and the other as γ_2, see Figure 1.2. Assuming we can retract γ_1 to γ_2 (there are no singularities in between those two paths), we can use the Stokes theorem to get

$$\int_{\gamma_1} A - \int_{\gamma_2} A = \oint_\gamma A = \int_\Sigma F = 0 \,. \tag{1.21}$$

We can deform the path in the Wilson loop as far as no singularity is crossed, and the expectation value of the Wilson loop would be the same. It is in this sense that Wilson loops are topological in CS theory. Once the theory is quantised, a prominent role is played by the expectation value of the Wilson loop. It is this observable that represents the previously mentioned knot invariant. It was shown by Witten in [41] that this expectation value corresponds

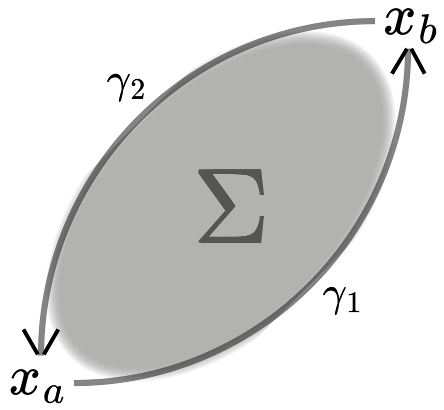

Figure 1.2: A figure illustrating the topological nature of Wilson loops in Abelian CS theory.

to the Jones polynomial for the knot, once an appropriate identification between the CS level for the gauge group $SU(2)$ in the fundamental representation and the Jones polynomial variables is made. Generalizing this to the more general rank of the gauge group and more general representations, we arrive at the colored HOMPFLY-PT polynomials[1].

On a more mathematical side, we can make the following comment [43]. For a given group G and a trivial G-bundle and two connections on this bundle A and $\tilde{A}$, it can be shown (Chern-Weil theorem) that

$$\left\langle \tilde{F}, ...\tilde{F} \right\rangle_n - \left\langle F, ..., F \right\rangle_n = \mathrm{d}Q(\tilde{A}, A) \;, \tag{1.22}$$

where

$$Q(\tilde{A}, A) = n \int_0^1 \mathrm{d}t \left\langle \tilde{A} - A, F_t, ..., F_t \right\rangle_n . \tag{1.23}$$

In this formula

$$F_t = \mathrm{d}A_t + A_t^2 \,, \qquad\qquad A_t = A + t\left(\tilde{A} - A\right) \,, \tag{1.24}$$

so that A_t is a homotopic connection interpolating between A and $\tilde{A}$. Furthermore, $\left\langle \dots \right\rangle$ stands for an invariant symmetric tensor for G. As the right-hand side of (1.22) is manifestly symmetric, we can take $\left\langle \dots \right\rangle$ to be the trace in a concrete representation of the gauge algebra, due to the cyclicity of the trace. However, note that this is not necessary, and there are examples where it is fruitful to consider more general symmetric invariant tensors than those resulting from traces [44]. By taking $A = 0$ (which can be done on a trivial G-bundle) and relabeling $\tilde{A}$ with A, we obtain

$$Q(A)_{CS} = n \int_0^1 \mathrm{d}t \left\langle A, F_t, ..., F_t \right\rangle_n \,, \tag{1.25}$$

where now

$$F_t = t\mathrm{d}A + t^2 A^2 = tF + (t^2 - t)A^2 \,. \tag{1.26}$$

Equation (1.25) defines the CS form and CS action in any odd number of spacetime dimensions. As the primary example of higher-dimensional CS theory we will be interested in this thesis is $5D$ CS theory, we explicitly compute its form. From (1.25), the action is given by

$$\int \left\langle F^2 A - \frac{1}{2} F A^3 + \frac{1}{10} A^5 \right\rangle \,, \tag{1.27}$$

[1]The author of this PhD thesis did work on HOMFLY-PT polynomials in [42].

Equations of motion are of the form $\langle T_a F^2 \rangle = 0$, and they do not imply that the gauge curvature vanishes. However, $F = 0$ is a solution of the equations and a degenerate one. If we try to do a perturbation theory around $F = 0$ by inserting $A = A_{\text{flat}} + \delta A$ and expanding to the first order in δA, we would get that the equations are automatically satisfied. Again, it is customary to, instead of working with some general $\Big\langle \dots \Big\rangle$, work in a concrete representation of a gauge group where $\Big\langle \dots \Big\rangle$ is substituted with the trace in the representation. CS gauge theory is particularly suitable for formulating noncommutative models of quantum gravity, and in the next subsection, we explain the motivation for studying NC gravity.

1.2.3 Noncommutative gravity

At small scales, the smooth structure of a spacetime is believed to be broken. One way to model this is through the use of noncommutative geometry. Classically, coordinates are commuting numbers. In NC theory, space-time coordinates have nontrivial commutation relations. Heuristically, NC can be motivated as follows [45]. If we try to zoom in on some space-time region, we have to import a large amount of energy. This can create a black hole, and we are unable to see anything beyond its horizon, implying some sort of uncertainty relation

$$\Delta\hat{x}^\mu \Delta\hat{x}^\nu \sim |\theta^{\mu\nu}(\hat{x})| \ . \tag{1.28}$$

Formal way to ensure this is to postulate that the coordinates do not commute, with

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}(\hat{x}) \ . \tag{1.29}$$

There are multiple ways to proceed from this starting point. One particular way that is well-studied is to consider a field theory (scalar, spinor, gauge theory,...) defined on a plane with $\theta^{\mu\nu}$ being a constant antisymmetric matrix. In this case, NC effects can be introduced by the Moyal product

$$f \star g = e^{\frac{i}{2}\theta^{\mu\nu}\partial_{x^\mu}\partial_{y^\nu}} f(x)g(y)|_{x\to y} \,. \tag{1.30}$$

It follows directly from this definition that, if we consider coordinates as functions, we have

$$x^\mu \star x^\nu - x^\nu \star x^\mu = x^\mu x^\nu + \frac{i}{2}\theta^{\mu\nu} - x^\nu x^\mu - \frac{i}{2}\theta^{\nu\mu} = i\theta^{\mu\nu} \,. \tag{1.31}$$

Morally speaking, one way to pursue this program is to substitute all the products in the classical action with a star product and to substitute classical fields with their NC analogue. More details on this will be shown in Chapter 7. We emphasise that this is not the only way to introduce NC geometry and to analyse NC spacetime physics. As the primary goal of this thesis is to discuss considerations along the line of gauge/gravity duality (which, for the purpose of this thesis, is a synonym for the holographic duality), one goal is to use this effective approach to quantum gravity as a deformation of a classical bulk. As we shall see later, it is much simpler to study QFT on NC spacetime than to define a NC general relativity; luckily, the gauge-theoretic formulation of gravity can be used, and all the methodology from field theory on a NC spacetime can be applied. Additionally, instead of deforming the algebra of functions on a spacetime, one can represent coordinates as operators acting on a certain vector space. Of course, in this thesis, we will be interested in those approaches where the first-order formulation of gravity has an important role.

In the context of holographic duality, NC spacetimes are far from being well-studied as models of a bulk "spacetime". Singular examples include [46] and more recently [47]. The latter work is motivated by the double-skaling limit of the SYK model, which is a concrete quantum mechanical model of N Majorana fermions with random interaction. It turns out that in this limit, the NC bulk picture naturally emerges from chord diagrams. Boundary correlation functions have q-deformed symmetry, which is a quantum group type of symmetry usually encountered when studying NC spacetime.

1.2.4 Quantum-geometry relation

This subsection is based on author's work [48]

Classical gravity is perhaps the first theory that heavily relies on the study of manifolds. However, it is by no means the only one. The purpose of this subsection is to demonstrate that certain aspects of quantum mechanics can be modeled using manifolds. If AdS/CFT is true, this should indeed be correct, and many quantum-mechanical properties should be expressed in the bulk as geometric quantities. The most famous example of this correspondence is the Ruy-Takayanagi proposal for computing the entanglement entropy, which is introduced in Chapter 3 of this thesis.

While it may not be obvious at first, manifolds play an important role in QFT. The particular cases of QFTs, where this is manifest, are topological quantum field theories (TQFTs). Generically, TQFT is a monoidal functor between the category of cobordisms and the category of vector spaces. Roughly speaking, cobordism is a manifold representing the transition from the initial state of the system to the final one. TQFTs have the characteristic that the transition amplitude depends only on the topology of the spacetime and not on the additional details [49]. Very often, the cobordism is depicted using figures, so that many quantum mechanical properties of the system can be represented graphically [50]. Quantum mechanics can be viewed as a $0+1$ dimensional QFT, and there are many proposals for the graphical languages exploring different aspects of the theory. Of course, this does not support or disprove the AdS/CFT; it is completely unrelated to this proposal, but it still contributes toward the claim that geometry might encode some quantum properties. In [48], we considered a particular realization of the graphical language needed to geometrically interpret certain quantum protocols: quantum teleportation, entanglement swapping, and superdense coding. Quantum teleportation protocol is depicted in Figure 1.3. We shall not spend time here to explain the graphical language; an interested reader can consult [48]. Instead, we will continue our goal to learn something about the QFT using (Riemann-Cartan and NC) geometry using the bottom-up AdS/CFT correspondence.

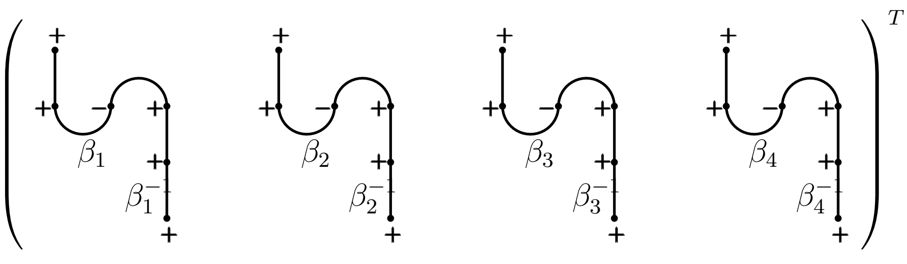

Figure 1.3: Quantum teleportation protocol, using the graphical language discussed in the text.

1.3 AdS spacetime and holography

We come to the point where we should discuss the holographic duality. Broadly speaking, this is an idea that quantum gravity can be explained in terms of an ordinary QFT in one dimension lower. The most well-understood example of holographic duality is AdS/CFT [3], which relates gravity (and other fields) on asymptotically *AdS* spacetime to conformal field theory (CFT) observables. Here, we elaborate on why *AdS* spacetime, the main character in our work, is so special. For the purpose of this thesis, *AdS* spacetime can be considered as a spacetime with a line element

$$\mathrm{d}s^2 = \ell^2 \frac{\mathrm{d}z^2 - \mathrm{d}t^2 + \sum_{i=2}^{D-1} \mathrm{d}x_i \mathrm{d}x_i}{z^2} \tag{1.32}$$

Coordinate z plays an important role in holographic duality, and $z = 0$ is the asymptotic, *conformal* boundary of this spacetime. At each $z =$ const. hypersurface, we have induced a Minkowski metric, with scaling factor $\frac{1}{z}$ growing as we approach this conformal boundary. Of course, the particular value of the coordinate $z = 0$ does not correspond to points in this manifold, and therefore this is not a boundary of spacetime in the same sense as a bottle is a boundary of the fluid inside it. We will refer to the limit $z \to 0$ as the boundary limit in the rest of the thesis. Coordinates (1.32) are called the Poincaré coordinates, and they do not cover the whole *AdS* spacetime in Lorentzian signature.

More generally, *AdS* can be defined as a hyperboloid embedded in the surrounding flat spacetime with signature $(-, +, \ldots, +, -)$. The line element of this flat spacetime is given by

$$\mathrm{d}s^2 = -\mathrm{d}V^2 - \mathrm{d}U^2 + \sum_{i=1}^{D-1} \mathrm{d}X_i^2 \,. \tag{1.33}$$

An embedded hyperboloid is defined through the relation

$$-V^2 - U^2 + \sum_{i=1}^{D-1} (X_i)^2 = -\ell^2, \tag{1.34}$$

where ℓ is the *AdS* radius. For simplicity, we focus from now on to the case of AdS_5. Define new coordinates that parametrize this embedded spacetime as

$$\begin{aligned} V &= \ell \frac{\cos(t/\ell)}{\cos\rho}\,, & X_1 &= \ell \,\mathrm{tg}\,\rho\,, \\ X_2 &= \ell \,\mathrm{tg}\,\rho \cos\psi\,, & X_3 &= \ell \,\mathrm{tg}\,\rho \sin\psi \cos\theta\,, \\ X_4 &= \ell \,\mathrm{tg}\,\rho \sin\psi \sin\theta\,, & U &= \ell \frac{\sin(t/\ell)}{\cos\rho}\,. \end{aligned} \tag{1.35}$$

It is obvious that the above relations are periodic in t under $t \to t + 2\pi\ell$. In order to avoid this time identification, we shall unfold the time coordinate and refer to the resulting spacetime as the *AdS* spacetime in the following (though, technically speaking, this is the universal cover of the *AdS* spacetime). Using (1.35), the line element takes the form

$$\mathrm{d}s^2 = \frac{\ell^2}{\cos^2\rho} \left(-\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sin^2\rho(\mathrm{d}\psi^2 + \sin^2\psi(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2))\right) \tag{1.36}$$

Another set of useful coordinates is obtained by taking $\rho = z^2$, such that the metric takes the form of

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{4\rho} + \frac{1}{\rho} \eta_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta \,. \tag{1.37}$$

This form of metric, and this type of coordinates, will be referred to as the *Fefferman-Graham* coordinates [51].

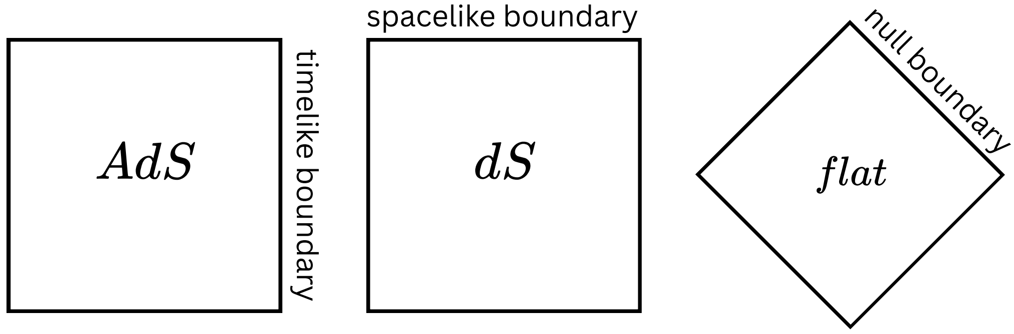

Figure 1.4: Penrose diagrams for two-dimensional AdS, dS, and Minkowski spacetime, with appropriate boundaries that could be used to analyze holographic duality.

Let us now consider two other maximally symmetric solutions of Einstein's equations: de Sitter (dS) spacetime and Minkowski spacetime. Unlike AdS, dS has no spatial boundary. In suitable coordinates, the metric of this spacetime can be written as

$$\mathrm{d}s^2 = \frac{-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2}{\eta^2} \, . \tag{1.38}$$

The limit $\eta \to 0$ can be considered as a boundary, called in the inflatory scenario the (*reheating surface*); however, this boundary is spacelike. There have been proposals to define dS/CFT correspondence using this surface, though this necessarily implies that the dual CFT is in Euclidean signature [52]. There exist other proposals for dS/CFT duality, but it is fair to say that they are more complicated than the AdS/CFT duality. Similarly, we shall discuss the Minkowski spacetime. This flat solution of Einstein's vacuum equations has a spatial boundary that is not particularly interesting for holographic duality. On the other hand, the null-boundary of a flat spacetime plays an important role, and it is on this boundary that one can realise a bottom-up version of holographic duality (flat holography [53]). Depending on the technical details, there exist celestial holography and Carlolian holography. Both are, however, significantly less understood than the AdS/CFT duality.

Chapter 2

Riemann-Cartan spacetime and CS gravity

In this chapter, we will introduce the basic methodological tools used in this thesis. As our primary goal is to understand the holographic properties of Riemann-Cartan spacetimes, we shall first explain in depth the formalism used to address those spaces, and then provide some examples of gravity theories relevant to our work. One can consult Appendix A.1 for the conventions. Note that the wedge product between forms is implied and omitted in most places where this does not introduce confusion.

2.1 First-order formalism

The basic field in Einstein's general relativity is the metric field $g_{\mu\nu}$. There is a unique metric-compatible torsion-free connection derived from this field, the Christoffel symbols $\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right)$. Riemann curvature $R^{\mu}{}_{\nu\rho\sigma}$ is defined from this object as

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\mu}_{\lambda\rho} - \Gamma^{\lambda}_{\nu\rho}\Gamma^{\mu}_{\lambda\sigma}\,, \tag{2.1}$$

and involves second derivatives of the metric field. Einstein equations are derived from the Einstein-Hilbert action

$$\frac{1}{16\pi G}\int \mathrm{d}^D x\,\sqrt{-g}R\,. \tag{2.2}$$

However, a metric tensor can be put into the form of the Minkowski metric at each individual point. We define

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}\,, \tag{2.3}$$

where the change of basis is effected by a matrix whose elements e^a_μ will be called *vielbeins*. We can further define a one-form $e^a = e^a_\mu \mathrm{d}x^\mu$, which we also call vielbein, if no confusion in the terminology arises. From the inverse e^μ_a we can create vector fields $\mathsf{e}_a = e^\mu_a\,\partial_\mu$, that we call the frame fields. Sometimes it is useful to adopt matrix notation and define a matrix e whose components are the coefficients e^μ_a. As this matrix is not necessarily symmetric, it can be useful to separate the first and the second index as $e_a{}^\mu$. We shall be dealing mostly with symmetris vielbeins, and therefore, for convenience , we will stick to the notation e^μ_a. Having in mind the importance of the vielbeins in this thesis, we proceed to analyse two important examples.

Let us consider a two-dimensional AdS spacetime in Poincaré coordinates (1.32). This means that the line element is given by

$$\mathrm{d}s^2 = \frac{-\mathrm{d}t^2 + \mathrm{d}z^2}{z^2} \,. \tag{2.4}$$

The change of basis to the orthogonal, Minkowski one, is performed as

$$g = \begin{pmatrix} -\frac{1}{z^2} & 0 \\ 0 & \frac{1}{z^2} \end{pmatrix} = \mathsf{e} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathsf{e}^T \,,$$

such that the frame fields are given by $\mathsf{e}_z = z\partial_z$ and $\mathsf{e}_t = z\partial_t$. It is simple to note that

$$e_a^\mu(x) = \mathsf{e}_a x^\mu \,, \tag{2.5}$$

which follows from considering the coordinate x^μ as a scalar function. In our example, we have

$$\mathsf{e}_z(z) = z, \quad \mathsf{e}_t(t) = z, \quad \mathsf{e}_z(t) = 0, \quad \mathsf{e}_t(z) = 0 \,. \tag{2.6}$$

As for the vielbein one form, they are given by $e^t = \frac{1}{z}\mathrm{d}t$ and $e^z = \frac{1}{z}\mathrm{d}z$. The second example, which we will also need for future reference, is the analysis of a torsionless (four-dimensional, for concreteness) black hole. Starting from the metric[1]

$$\mathrm{d}s^2 = -f^2(r)\mathrm{d}t^2 + \frac{1}{f^2(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \,, \tag{2.7}$$

we can choose the vielbeins as

$$e^0 = f(r)\mathrm{d}t, \quad e^1 = \frac{\mathrm{d}r}{f(r)}, \quad e^2 = r\mathrm{d}\theta, \quad e^3 = r\sin\theta\mathrm{d}\phi \,. \tag{2.8}$$

Obviously, vielbeins play a role similar to the metric field in the second-order formulation of gravity. However, in the first-order formulation of gravity, it is not assumed that the connection is determined from the metric, and the role of a connection is taken by a new field: the spin-connection ω^{ab}. Only if the geometry is assumed to be Riemannian (or if this follows from the equations of motion), the spin-connection is related to the vielbeins through the relation $\mathrm{d}e^a + \omega^{ab}e_b = 0$. Using this one-form, we can define the Riemann-Cartan curvature two-form as

$$R^{ab} = \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \equiv \omega^{ab} + \omega^a{}_c\omega^{cb} \,. \tag{2.9}$$

From the condition that the geometry is Riemannian, the nonzero components of the spin-connection are given by

$$\begin{aligned} \omega^{01} &= \frac{1}{2}(f^2(r))'\mathrm{d}t, \quad \omega^{12} = -f(r)\mathrm{d}\theta, \\ \omega^{13} &= -f(r)\sin\theta\mathrm{d}\phi, \quad \omega^{23} = -\cos\theta\mathrm{d}\phi. \end{aligned} \tag{2.10}$$

Riemann curvature is computed to be

$$\begin{aligned} R^{01} &= -\frac{1}{2}(f^2)''e^0e^1, \quad R^{02} = -\frac{1}{2r}(f^2)'e^0e^2, \\ R^{03} &= -\frac{1}{2r}(f^2)'e^0e^3, \quad R^{12} = -\frac{1}{2r}(f^2)'e^1e^2, \\ R^{13} &= -\frac{1}{2r}(f^2)'e^1e^3, \quad R^{23} = -\frac{1}{2}(f^2)''e^2e^3. \end{aligned} \tag{2.11}$$

[1] For convenience, we changed notation $f \to f^2$ from (1.4).

Previous formulas can be applied to any four-dimensional Riemannian BH geometry, as is the case of four-dimensional Schwarzschild AdS black hole, with the line element given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r} + \frac{r^2}{\ell^2}\right)\mathrm{d}t^2 + \frac{1}{\left(1 - \frac{2m}{r} + \frac{r^2}{\ell^2}\right)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,. \tag{2.12}$$

EH action in the first-order formalism can be written as

$$\frac{1}{32\pi G_N}\int \varepsilon_{abcd}R^{ab}e^c e^d\,. \tag{2.13}$$

Let us prove that this is true, as we will need similar computations multiple times in this thesis.

$$\begin{aligned}
\int \varepsilon_{abcd}R^{ab}e^c e^d &= \int \varepsilon_{abcd}\frac{1}{2}R^{ab}{}_{\mu\nu}e^c{}_\rho e^d{}_\sigma \mathrm{d}x^\mu \mathrm{d}x^\nu \mathrm{d}x^\rho \mathrm{d}x^\sigma \\
&= -\int \frac{1}{2}\varepsilon_{abcd}\varepsilon^{\mu\nu\rho\sigma}R^{\lambda\tau}{}_{\mu\nu}e^a{}_\lambda e^b{}_\tau e^c{}_\rho e^d{}_\sigma \mathrm{d}^4x \\
&= -\int \frac{1}{2}eR^{\lambda\tau}{}_{\mu\nu}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\lambda\tau\rho\sigma}\mathrm{d}^4x \\
&= \int eR^{\lambda\tau}{}_{\mu\nu}\left(\delta^\mu_\lambda\delta^\nu_\tau - \delta^\mu_\tau\delta^\nu_\lambda\right)\mathrm{d}^4x \\
&= 2\int \mathrm{d}^4x\,\sqrt{-g}R
\end{aligned} \tag{2.14}$$

where $e \equiv \det(e^a_\mu) \equiv \sqrt{-g}$. In a similar fashion, the cosmological constant term, in the first-order formalism, can be written as

$$\begin{aligned}
\int \varepsilon_{abcd}\,e^a e^b e^c e^d = -\int \varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho e^d{}_\sigma \varepsilon^{\mu\nu\rho\sigma}\mathrm{d}^4x &= -\int \sqrt{-g}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\mathrm{d}^4x \\
&= 24\int \mathrm{d}^4x\sqrt{-g}\,.
\end{aligned} \tag{2.15}$$

Therefore, a four-dimensional EH action with cosmological constant can be written as

$$\frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-g}\,(R - 2\Lambda) = \frac{1}{32\pi G_N}\int \varepsilon_{abcd}\left(R^{ab} - \frac{\Lambda}{6}e^a e^b\right)e^c e^d\,. \tag{2.16}$$

First-order formalism naturally incorporates the torsion tensor. We define a two-form T^a such that

$$T^a = \mathrm{d}e^a + \omega^a{}_b e^b\,, \tag{2.17}$$

or, in full-coordinate coordinate indices $T^\mu_{\nu\rho} = e^\mu_a T^a_{\nu\rho}$. Torsion is antisymmetric in the two lower indices $T^\mu_{\nu\rho} = -T^\mu_{\rho\nu}$. A more traditional definition of torsion is the antisymmetric part of the affine connection

$$T^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} - \Gamma^\mu_{\rho\nu}\,. \tag{2.18}$$

Throughout the thesis, we will often use the identities

$$\mathrm{D}R^{ab} = 0\,, \qquad \mathrm{D}T^a = R^a{}_b e^b\,. \tag{2.19}$$

Having introduced the technical details of the first-order gravity formulation and torsion, in the next section, we provide an overview of the use of torsion in different areas of physics.

2.2 Overview of torsion in high-energy and low-energy physics

There are various opinions in the scientific community on whether the torsion field plays an important role in gravity. The range of claims goes from the opinion that torsion $T^{\rho}_{\mu\nu}$ is just an additional tensor field, similar to other matter fields that one can anyway consider in general relativity, and thus is not very interesting to consider it (one example of such a claim can be found in Carroll's book on general relativity [54]), to the claims that torsion, as part of geometry that is independent of the metric field, plays an essential role in our understanding of classical gravity. A comprehensive discussion, arguing against the claims that torsion is irrelevant, can be found in [55]. Another negative attitude towards torsion is the claim that torsion violates the equivalence principle. This is usually argued as follows. Locally, in GR, we can choose a coordinate system such that the components of the metric field at a given point coincide with the flat metric $\eta^{ab} = \text{diag}(-1, +1, +1, \dots, +1)$ and such that the components of the connection vanish at that point. If the connection is not symmetric, no coordinate transformation will make all the components equal to zero at a given point, and thus, one may claim that the equivalence principle is broken. However, the implicit assumption in this claim is that we should only care about coordinate transformations: the spin-connection ω^{ab}_{μ} has two Lorentz indices, and one can act on them with an arbitrary local Lorentz transformation. It turns out that in this way, one can indeed make the spacetime with torsion locally equivalent to the Minkowski spacetime and thus prove the equivalence principle [34]. Therefore, we can work with Riemann-Cartan spacetime without having to sacrifice something as important as the equivalence principle. Indeed, there is an abundant amount of work dedicated to understanding the role of tosrion in our universe: from cosmology [56] to black holes [57] and more. However, no experimental observation of torsion in our universe has been made so far, and thus a natural question we ask is if our understanding of torsion can be applied to address certain aspects aside from four-dimensional gravity phenomenology.

Before we proceed, note that the common interpretation of torsion is that, starting from a point with coordinates x^{μ} and taking a parallel transport along two distinct directions, the final point will differ depending on the order of the parallel transports, see Figure 2.1.

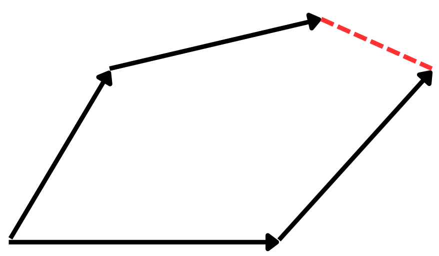

Figure 2.1: Schematic representation of torsion as a failure of closing a parallelogram.

In condensed matter physics, torsion is usually studied in order to introduce dislocations in the system [58, 59]. One way to understand this is by noting, as in figure 2.1, that torsion corresponds to the failure of closing a parallelogram when parallel transporting along two vectors. On the other hand, we can imagine a crystal structure as given in figure 2.2, with a half-line of sites removed. Then, it is obvious that starting from one site, as in the figure, and going three unit steps in one direction, then three unit steps in the normal direction, and so on, we end up in a different site than the one we initially started in. This discrepancy (or, more precisely, the holonomy of translations) is given by the Burgers vector, and in the continuum limit, is related to the torsion tensor. We therefore believe that the knowledge of torsion, acquired by studying high-energy physics, should be applied to condensed-matter systems, with eventual potential applications in the laboratory. Gauge/gravity duality offers one possible way of combining the high-energy gravity physics and condensed-matter phenomenology, and we will analyse the influence of torsion on condensed-matter systems holographically in Chapter 6. In the next section, we will introduce the CS gravity and see how torsion plays an important role in this theory.

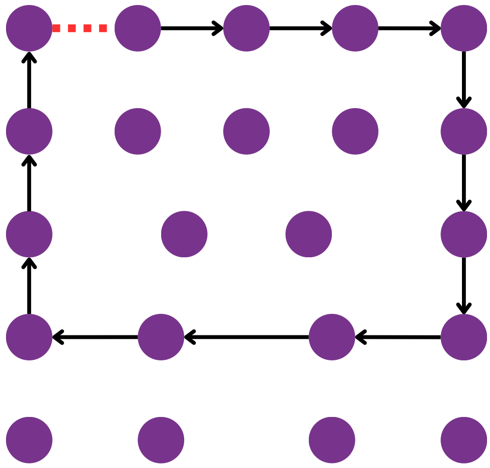

Figure 2.2: $2D$ latice with dislocations. Starting from a certain position, a semi-infinite line of lattice sites is removed. Following the lines as in the figure, we see that starting from one point and going around the defect by making three steps in each direction, we end up in a different site than where we initially started, thus making the analogy with torsion manifest.

2.3 Chern-Simons gravity

Let us start from $D = 3$ EH gravity without matter. As usual, the equations of motion imply that the spacetime is on-shell Ricci flat, $R_{\mu\nu} = 0$. However, in three spacetime dimensions, there is a kinematic relation between $R_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$, namely

$$R_{\mu\nu\rho\sigma} = R_{\nu\sigma}g_{\mu\rho} + R_{\mu\rho}g_{\nu\sigma} - R_{\nu\rho}g_{\mu\sigma} - R_{\mu\sigma}g_{\nu\rho} - \frac{1}{2}R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\,. \tag{2.20}$$

Therefore, on-shell, Riemann curvature vanishes, and every solution is a locally flat (Minkowski) spacetime. This further implies that there is no local dynamics in $3D$ gravity, spacetime remains still, and no gravitational waves can be produced. A similar conclusion holds once the cosmological constant is introduced, and on-shell, the spacetime is locally (A)dS.

At this point, we can see the similarity between action (1.17) and $3D$ EH gravity: in the former case, the gauge curvature vanishes on-shell, and in the latter, the curvature of spacetime vanishes. Actually, this correspondence can be made more formal by computing the gauge curvature of (1.16). This derivation can be applied to any number of spacetime dimensions, as it relies only on the commutational relations among the generators of the AdS_D algebra, so we will be general at this point. Denoting generators as J_{ab} and $P_a \equiv J_{aD}$, the commutation relations take the form

$$[J_{AB}, J_{CD}] = G_{AD}J_{BC} + G_{BC}J_{AD} - (C \leftrightarrow D)\ , \tag{2.21}$$

where $A = (a, D)$ and $G_{AB} = \mathrm{diag}(-, +, \ldots, +, -)$. An explicit computation gives

$$\begin{aligned} F &= \mathrm{d}A + A \wedge A = \mathrm{d}A + \frac{1}{2}[A, A] \\ &= \frac{1}{2}\mathrm{d}\omega^{ab}J_{ab} + \frac{1}{\ell}\mathrm{d}e^a J_{aD} + \frac{1}{2}\omega^{ab}\omega^{cd}G_{ad}J_{bc} + \frac{1}{\ell}\omega^{ab}e^c G_{bc}J_{aD} - \frac{1}{\ell^2}e^a e^b J_{ab} \\ &= \frac{1}{2}\left(\mathrm{d}\omega^{ab} + \omega^a{}_c\omega^{cb} + \frac{1}{\ell^2}e^a e^b\right) J_{ab} + \frac{1}{\ell}\left(\mathrm{d}e^a + \omega^a{}_b e^b\right) J_{aD} \\ &= \frac{1}{2}\left(R^{ab} + \frac{1}{\ell^2}e^a e^b\right) J_{ab} + \frac{1}{\ell}\mathrm{D}e^a J_{aD} \\ &= \frac{1}{2}\left(R^{ab} + \frac{1}{\ell^2}e^a e^b\right) J_{ab} + \frac{1}{\ell}T^a P_a\ . \end{aligned} \tag{2.22}$$

Taking $D = 3$, we observe that the equations of motion of $3D$ EH gravity with negative cosmological constant $\Lambda = -\frac{1}{\ell^2}$ are realized by the condition $F = 0$. In a sense, a classical equivalence exists between three-dimensional EH gravity and CS theory for (a noncompact) $SO(2, 2)$ gauge group. We emphasize the word classical, as one encounters problems when trying to use CS theory as a quantum theory of gravity [60, 61]. Namely, the invertibility of the vielbein is an important condition as it leads to the nondegeneracy of the metric tensor. While classically we can make sure that this holds by putting this condition by hand, in a quantum theory, we have to modify the path integral such that we do not integrate over those configurations. In turn, this may drastically change the theory.

To be more concrete, we will try to connect CS theory and EH gravity in three dimensions directly. The most straightforward way is to define an invariant tensor of rank two,

$$\langle J_{ab}, P_c\rangle = \varepsilon_{abc}\ , \tag{2.23}$$

with all other components equal to zero. Furthermore, because of the Chern-Weil theorem (1.22), we start by computing $\langle FF\rangle$:

$$\begin{aligned} \int_{\mathcal{M}_4}\langle FF\rangle &= \int_{\mathcal{M}_4}\left\langle\left[\frac{1}{2}\left(R^{ab} + \frac{1}{\ell^2}e^a e^b\right) J_{ab} + \frac{1}{\ell}T^a P_a\right]\left[\frac{1}{2}\left(R^{cd} + \frac{1}{\ell^2}e^c e^d\right) J_{cd} + \frac{1}{\ell}T^c P_c\right]\right\rangle \\ &= \int_{\mathcal{M}_4}\left(R^{ab} + \frac{1}{\ell^2}e^a e^b\right)\frac{1}{\ell}T^c\langle J_{ab}, P_c\rangle = \int_{\mathcal{M}_4}\varepsilon_{abc}\frac{1}{\ell}\left(R^{ab} + \frac{1}{\ell^2}e^a e^b\right)\mathrm{D}e^c \\ &= \frac{1}{\ell}\int_{\mathcal{M}_4}\mathrm{d}\left[\varepsilon_{abc}\left(R^{ab}e^2 + \frac{1}{3\ell^2}e^a e^b e^c\right)\right]\ , \end{aligned} \tag{2.24}$$

from which, by using the Stokes theorem, it follows that the CS Lagrangian takes the form

$$k\int_{\mathcal{M}_3}\varepsilon_{abc}\left(R^{ab}e^c + \frac{1}{3\ell^2}e^a e^b e^c\right)\ . \tag{2.25}$$

This indeed is the action for the EH gravity with a negative cosmological constant in three spacetime dimensions. Note that in three dimensions, it is sometimes useful to introduce

$$\omega^a \equiv \frac{1}{2}\epsilon^{abc}\omega_{bc}\,. \tag{2.26}$$

Additionally, we may work with a concrete representation of the gauge group and take the invariant symmetric tensor to be given by the trace in this representation; we will use this logic in the next subsection. Here, we further wish to show an alternative approach that is valid in three dimensions and relies on the isomorphism $\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, where we also have $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(2,1)$. Algebra $\mathfrak{sl}(2,\mathbb{R})$ is generated by three elements, commonly denoted as H, E_+ and E_-. They satisfy the following commutational relations

$$[H, E_+] = E_+ \,, \quad [H, E_-] = -E_- \,, \quad [E_+, E_-] = 2H \,. \tag{2.27}$$

On the other hand, generators of algebra $\mathfrak{so}(2,1)$ are standardly denoted by J_a, and they satisfy the commutation relations $[J_a, J_b] = \varepsilon_{ab}{}^c J_c$, where the upper and lower indices follow the usual logic in the Lorentzian signature. If we define $\mathfrak{so}(2,1)$ gauge connections as

$$A = \left(\omega^a + \frac{1}{\ell} e^a\right) J_a^+ \,, \quad \overline{A} = \left(\omega^a - \frac{1}{\ell} e^a\right) J_a^- \,, \tag{2.28}$$

where $J_a^\pm$ are generators of the two copies of $\mathfrak{so}(2,1)$ in the $\mathfrak{so}(2,2)$ algebra. For a concrete representation of this algebra, consult section 4.4, as for now, we note that the trace we shall use to define the CS action can be choosen to satisfy $\mathrm{Tr}(J_a J_b) = \frac{1}{2}\delta_{ab}$ and $\mathrm{Tr}(J_a J_b J_c) = \frac{1}{4}\varepsilon_{abc}$. The action for three-dimensional gravity can then be written, modulo boundary terms, as

$$2k \int \mathrm{CS}(A) - 2k \int \mathrm{CS}(\overline{A}) \,, \tag{2.29}$$

where $CS(A)$ is the CS top form from (1.17), see for example [62]. It is common to set $\ell = 1$ when writing the gravity action, and in the following, very often we will employ this.

2.3.1 Action for 5D CS gravity and 3D MB model

In this section, we take a concrete representation for the gauge group $SO(4,2)$ and compute the action for CS gravity. For a comprehensive presentation of CS gravity, see [33]. Lie group $SO(4,2)$ is the AdS group in five dimensions, and we denote its generators, as before, by J_{AB}. Lie brackets are given by (2.21) where now, for concreteness, we have $G_{AB} = (-++++-)$. By making the decomposition of generators into J_{ab} and $J_{a5} \equiv P_a$, where the indices take the value $(a, b = 0, 1, 2, 3, 4)$, we obtain the following form of Lie brackets for $\mathfrak{so}(4,2)$

$$\begin{aligned}
[J_{ab}, J_{cd}] &= G_{ad} J_{bc} + G_{bc} J_{ad} - G_{ac} J_{bd} - G_{bd} J_{ac} \,, \\
[J_{ab}, P_c] &= G_{bc} P_a - G_{ac} P_b \,, \\
[P_a, P_b] &= P_{ab} \,,
\end{aligned} \tag{2.30}$$

where $G_{ab} = (-, +, +, +, +)$. This is the general form of the AdS Lie algebra in any dimension, and we can use the form of gauge connection in (1.16). We shall be working in a concrete representation of this algebra, defined using the Gamma matrices, details of which are presented in Appendix A.1. Again, starting for conveniency from (1.22), rather than from (1.27), we have

$$\begin{aligned}
\mathrm{Tr}\left(F^3\right) =& \frac{1}{8} \mathrm{Tr}\left(J_{AB} J_{CS} J_{EF}\right) F^{AB} F^{CD} F^{EF} \\
=& \frac{1}{8} \mathrm{Tr}\left(J_{ab} J_{cd} J_{ef}\right) F^{ab} F^{cd} F^{ef} + \frac{6}{8} \mathrm{Tr}\left(J_{ab} J_{cd} J_{e5}\right) F^{ab} F^{cd} F^{e5} \\
&+ \frac{12}{8} \mathrm{Tr}\left(J_{ab} J_{c5} J_{e5}\right) F^{ab} F^{c5} F^{e5} + \frac{8}{8} \mathrm{Tr}\left(J_{a5} J_{c5} J_{e5}\right) F^{a5} F^{c5} F^{e5} \\
=& \frac{3i}{8} \varepsilon_{abcde} F^{ab} F^{cd} F^{e5} \,,
\end{aligned} \tag{2.31}$$

where all the traces are obtained from the formulas in Appendix A.1. Note that, as the *AdS* algebra takes the same form in any number of dimensions, we can use the results of (2.22) to compute the action $S^{(5)}_{CS}$.

$$\begin{aligned} S^{(5)}_{CS} &= \frac{-ik}{3}\int_{\mathcal{M}_6} \mathrm{Tr}\left(F^3\right) = \frac{k}{8}\int_{\mathcal{M}_6} \varepsilon_{abcde} F^{ab}F^{cd}F^{e5} \\ &= \frac{k}{8}\int_{\mathcal{M}_6} \varepsilon_{abcde}\left(R^{ab}+\frac{1}{\ell^2}e^e e^b\right)\left(R^{cd}+\frac{1}{\ell^2}e^c e^d\right)\mathrm{D}T^e \\ &= \frac{k}{8}\int_{\mathcal{N}} \varepsilon_{abcde}\left(\frac{1}{l}R^{ab}R^{cd}e^e + \frac{2}{3\ell^3}R^{ab}e^c e^d e^e + \frac{1}{5\ell^5}e^a e^b e^c e^d e^e\right). \end{aligned} \tag{2.32}$$

The equations of motion in the first-order formalism follow from variations in the vielbein and the spin-connection. This results in the following two sets of equations

$$\varepsilon_{abcde}\left(R^{ab}+e^a e^b\right)\left(R^{cd}+e^c e^d\right) = 0\,, \tag{2.33}$$

$$\varepsilon_{abcde}T^a\left(R^{bc}+e^b e^c\right) = 0\,. \tag{2.34}$$

Interestingly, the second equation is solved by $T^a = 0$, leaving us with only the first equation. In the case of $5D$ CS gravity, it is consistent to restrict oneself to the torsionless geometries, yet this is unnecessary, as there are, as we shall see later, solutions involving torsion. We can also cast the action (2.32) into a more familiar second-order form as

$$S^{(5)}_{CS} = \frac{k}{2\ell^3}\int \mathrm{d}^5x\sqrt{-g}\left[\frac{\ell^2}{4}\left(R^2 - 4R^{\mu\nu}R_{\nu\mu} + R^{\mu\nu\rho\sigma}R_{\rho\sigma\mu\nu}\right) + R + \frac{6}{\ell^2}\right], \tag{2.35}$$

where we were careful not to use the identity $R^{\mu\nu} = R_{\nu\mu}$, which is often used in Riemannian geometry, as it fails to be true on a Riemann-Cartan spacetime.

In three dimensions, we consider the following action

$$k\int \varepsilon_{abc}R^{ab}e^c - \frac{\Lambda}{3}\int \varepsilon_{abc}e^a e^b e^c + \alpha_3\int\left(\omega_a \mathrm{d}\omega^a + \frac{1}{3}\varepsilon_{abc}\,\omega^a\omega^b\omega^c\right) + \alpha_4\int T^a e_a\,, \tag{2.36}$$

where, apart from the standard EH term and the cosmological constant, we added the CS term for the spin-connection, and the translational CS term. This defines the Mielke-Baekler (MB) model [63] of topological gravity. The equations of motion for this model, obtained from (2.36) by varying with respect to e^a and ω^{ab} independently, are

$$\varepsilon_{abc}\left(k\,R^{bc} - \Lambda e^b e^c\right) + 2\alpha_4 T_a = 0\,, \tag{2.37}$$

$$k\,T^a + \varepsilon^a{}_{bc}\left(\alpha_3 R^{bc} + \alpha_4 e^b e^c\right) = 0\,. \tag{2.38}$$

Assuming $k^2 \neq \alpha_3\alpha_4$, this set of equations can be solved to get algebraic equations

$$R^{ab} = B\,e^a e^b\,, \qquad 2T^a = C\varepsilon^a{}_{bc}e^b e^c\,, \tag{2.39}$$

where

$$C = \frac{\alpha_3\Lambda + \alpha_4 k}{\alpha_3\alpha_4 - k^2}\,, \qquad B = -\frac{k\Lambda + \alpha_4^2}{\alpha_3\alpha_4 - k^2}\,. \tag{2.40}$$

It is then evident that the general form of spacetime is precisely the (A)dS with torsion determined from the vielbeins. We can further decompose the spin-connection into torsionless part and contorsion, resulting in

$$\omega^a \equiv \tilde{\omega}^a(e) + k^a = \tilde{\omega}^a(e) + \frac{C}{2}e^a\,. \tag{2.41}$$

This model constitutes the simplest testing ground for the relation between Riemann-Cartan bulk and physics at the asymptotic boundary [64, 65]. Importantly, this model can also be written as CS gravity theory, as shown in [66]. The novelty, compared to (2.28), is that the two gauge connections from (2.28) now take a slightly different form, resulting in

$$A = (\omega^a + q\, e^a)\, J_a^+ , \quad \overline{A} = (\omega^a + \overline{q}\, e^a)\, J_a^- , \tag{2.42}$$

where $q = \frac{1}{2}\left(-C + \sqrt{C^2 - 4B}\right)$ and $\overline{q} = \frac{1}{2}\left(-C - \sqrt{C^2 - 4B}\right)$. Furthermore, the CS level of two $\mathfrak{sl}(2, \mathbb{R})$ CS lagrangians is different, with the difference between two CS levels being proportional to the α_3 constant. In this thesis, we will mostly take a very simple choice of $\alpha_3 = 0$, so we refrain from making further comments on the $\alpha_3 \neq 0$ case here, and we refer the reader to [66, 26].

2.3.2 Boundary terms in first-order formalism

This section is partially based on the author's work [67]

The EH action (2.2) requires additional boundary terms if one wishes to consider manifolds with boundary. To see this, let us first analyse a much simpler model of classical mechanics, with action

$$-\int \mathrm{d}t\ \frac{1}{2} q \frac{\mathrm{d}^2 q}{\mathrm{d}t^2} . \tag{2.43}$$

Varying this action under $q(t) \to q(t) + \delta q(t)$ we obtain the equation of motion

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = 0 . \tag{2.44}$$

The same equation can be obtained starting from the commonly used action

$$\int \mathrm{d}t \frac{1}{2} \left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 , \tag{2.45}$$

and, indeed, we can show that those two actions are equal up to boundary terms. Namely, if we add the boundary term

$$\int_\partial \frac{1}{2} q \frac{\mathrm{d}q}{\mathrm{d}t} \equiv \frac{1}{2} q \frac{\mathrm{d}q}{\mathrm{d}t}\Big|_{t_1}^{t_2} \tag{2.46}$$

to the first action (2.43), it becomes equal to (2.45). However, variation of (2.43) results in a boundary term

$$-\int_\partial \frac{1}{2} q \delta \left(\frac{\mathrm{d}q}{\mathrm{d}t}\right) + \frac{1}{2} \int_\partial \frac{\mathrm{d}q}{\mathrm{d}t} \delta q, \tag{2.47}$$

which does not have to vanish under standard Dirichlet boundary conditions at the boundary: $\delta q(t_2) = \delta q(t_1) = 0$. The reason for this is that the Lagrangian in (2.43) explicitly depends on the acceleration $\frac{\mathrm{d}^2 q}{\mathrm{d}t^2}$, which is not usually present in the action. Once the boundary term (2.46) is added, this acceleration term is eliminated from the action. However, in field theory, and especially in gravity, one is often led to consider Lagrangians containing accelerations that can be eliminated using appropriate boundary terms. The simplest example is an action for a free scalar field, where instead of

$$S_1 = \frac{1}{2} \int \mathrm{d}^D x\ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi , \tag{2.48}$$

one considers

$$S_2 = -\frac{1}{2}\int \mathrm{d}^D x\, \phi\, \Box\phi\,. \tag{2.49}$$

Another significant example is EH action (2.2). Due to relation (2.1), Ricci scalar contains second derivatives of the metric field, and in this sense, we say that the EH action contains accelerations. We can add a boundary term to eliminate this acceleration, and this boundary term is called the Gibbons-Hawking-York (GHY) term. We can see the necessity for such a term by noting that the variation of EH action is given by

$$\delta S_{EH} = \int \mathrm{d}^D x \sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right)\delta g_{\mu\nu} + \int \mathrm{d}^D x \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\,. \tag{2.50}$$

By using the Palatini identity $\delta R_{\mu\nu} = \nabla_\alpha(\delta\Gamma^\alpha_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\alpha_{\mu\alpha})$, the second term in (2.50) is, using the Stokes theorem, given by the boundary term and thus is usually discarded when computing the equations of motion. However, the total action must be stationary, which is usually implemented by imposing Dirichlet boundary conditions on the fields. In this case, we should impose $\delta g_{\mu\nu}|_\partial = 0$. However, as evident from the previous formula, what enters the boundary term is the variation of the derivative of $g_{\mu\nu}$, and this object does not have to be zero if Dirichlet boundary conditions are satisfied. In order to eliminate this variation, an appropriate boundary term has to be added. This boundary term is again the Gibbons-Hawking-York term, derived by York in [68] and later used by Gibbons and Hawking in [69] to derive the BH entropy (see Chapter 1). The GHY term, for timelike boundaries, is given by the expression

$$S_{GHY} = \frac{1}{8\pi G}\int_{\partial\mathcal{N}} \mathrm{d}^d y\, \sqrt{|h|}K\,, \tag{2.51}$$

where K is the trace of the extrinsic curvature, y^α are coordinates on the boundary ($d = D-1$), and $h_{\alpha\beta}$ is the induced metric. If n^μ is a unit vector orthogonal to $\partial\mathcal{N}$, then this trace is defined to be $K = \nabla_\mu n^\mu$. For more details on the geometry of hypersurfaces, see [70]. Interestingly, this term is important for several reasons, as discussed in [71]. Apart from removing the boundary variation of the metric field, the GHY term also enables one to use the path integral formulation of gravity, provides the necessary boundary term in order to use the Hamiltonian formalism for gravity, so that one reproduces the ADM mass of a BH and gives the correct expression for the entropy of a BH when using the saddle-point approximation. Actually, in the case of Schwarzschild BH, the boundary term (2.51) is divergent and has to be regularized in a suitable manner. We shall encounter similar infinities once we start discussing the holographic duality.

It is, of course, impossible to give a simple answer to the question of what the boundary terms are in the first-order treatment of gravity, because the question itself is imprecise. We should first specify the quantity that we would like to compute, and only then ask ourselves what boundary term should be added. One example when boundary terms are important is when computing conserved charges. Conserved charges in the first-order formalism were thoroughly analysed in the past, probably culminating in the work of Nester [72, 73] and his famous formula for charges such as energy and angular momentum. This formula states that

$$Q_\xi = G\int_{S^2} \frac{1}{2}(\iota_\xi\omega^{ab})\Delta\rho_{ab} + \frac{1}{2}\Delta\omega^{ab}(\iota_\xi\overline{\rho}_{ab}) + (\iota_\xi e^a)\Delta\tau_a + \Delta e^a(\iota_\xi\tau_a)\,, \tag{2.52}$$

where $\rho_{ab} = \frac{\partial\mathcal{L}}{\partial R^{ab}}$ and $\tau_a = \frac{\partial\mathcal{L}}{\partial T^a}$ are covarinat momenta. Furthermore, ι_ξ denotes contraction with Killing vector field ξ, while $\Delta\rho_{ab} = \rho_{ab} - \overline{\rho}_{ab}$ being the difference between ρ_{ab} computed for a given spacetime and the background reference value $\overline{\rho}_{ab}$ computed for a spacetime wihout horizon. We can illustrate the usage of this formula for EH action (2.2) and Schwarzschild BH.

$$\rho_{ab} = \frac{\partial \mathcal{L}}{\partial R^{ab}} = \frac{1}{16\pi G}\varepsilon_{abcd}e^c e^d\,, \qquad \tau_a = 0\,. \tag{2.53}$$

Using the results (2.10), we can compute that the charge $Q_{\partial_t} \equiv E = M$, which, in turn, justifies the usage of M in (1.10) as the energy of the BH spacetime. Standardly, Nester's formula is derived using the Hamiltonian approach, which is not the main focus of this thesis, as we mostly work with the Lagrangian formulation. Even though one can say that in the Hamiltonian formulation, the issue of boundary terms in the first-order formulation of gravity is well-understood by the work of Nester, it is still puzzling how the first-order Lagrangian formalism, threating vielbein and spin-connection as independent fields, is different from the second-order formulation, because there is no analogue of (2.50). The EH Lagrangian in the first-order theory is given by (2.13). The latter contains only first derivatives in the fundamental field ω^{ab}, and thus there is no acceleration present in the Lagrangian. Variation with respect to ω^{ab} results in a boundary term

$$\int_{\mathcal{N}} \varepsilon_{abcd}\mathrm{D}\left(\delta\omega^{ab}\right)e^c e^d = \int_{\partial\mathcal{N}} \varepsilon_{abcd}\delta\omega^{ab}e^c e^d. \tag{2.54}$$

Putting Dirichlet boundary conditions on the field ω^{ab}, this term vanishes. One could then insist that in the first-order formulation of gravity, no boundary terms analogous to (2.51) are necessary in order to have a well-defined variational principle. However, the GHY term plays an important role in many different situations. It was, after all, originally used by Gibbons and Hawking to compute the BH entropy. How can we then rederive their result in the first-order gravity? It goes without saying that there are ways to derive the black hole's entropy in the first-order formalism. We already illustrated one way to do so: using the Nester formula to extract the energy and combining it with the first law of thermodynamics. A more systematic treatment, in which the first law of thermodynamics is derived, can be found in [74]; we shall use this result later in the thesis. Therefore, one could insist that no boundary terms should be a priori added to the gravity action in the first-order formalism if the action is linear in derivatives of fundamental fields. If we are dealing with the gauge theory of gravity, with a gauge connection of the form (1.16), it is even more evident that we should treat vielbein and spin-connection on equal footing and that it should be completely fine to put Dirichlet boundary conditions on ω^{ab}.

However, going through the literature, we can indeed find arguments that some version of the GYT term should also be added in the first-order formulation of gravity. In the case where the connection is derived from the metric, the variation of the connection at the boundary is precisely the reason why the appropriate GHY term should be supplemented. One logic then is to add the boundary term, which will cancel the boundary variation $\delta\omega^{ab}$ and move the variation to the vielbein. Such a term is given by

$$-\int_{\partial\mathcal{N}} \varepsilon_{abcd}\,\omega^{ab}e^c e^d\,. \tag{2.55}$$

Note that this boundary term explicitly involves the spin-connection, which is not a tensorial object. For this reason, we shall "covariantize" this expression, taking instead the term [75]

$$2\int_{\partial\mathcal{N}} \varepsilon_{abcd}\,n^a\mathrm{D}n^b e^c e^d\,. \tag{2.56}$$

Here, n^a is normal to the boundary, meaning that $e^a n_a|_{\partial\mathcal{N}} = 0$. The intuition for this covariant term is as follows. Note that $\mathrm{D}n^a = \mathrm{d}n^a + \omega^a{}_b n^b$. If we use coordinates as (2.7) (later we shall use another set of coordinates (4.58), which holds a similar property), the boundary is defined as $r = \text{const}$, so that the unit normal is proportional to $(0, 1, 0, 0)$. Because of this $\varepsilon_{abcd}\, n^a \mathrm{D}n^b e^c e^d \sim \varepsilon_{1bcd}\omega^{b1} e^c e^c$, which gives precisely the same result as (2.55) once inegrated over a constant r surface.

Fixing the vielbein instead of the spin-connection seems more natural from the perspective of BH thermodynamics. In this context, boundary conditions on bulk fields correspond to the choice of an ansamble on the thermodynamics side (for example, see [76, 77]). The simplest example is given by electrically charged black holes. Standard Yang-Mills term in the action is stationary provided the Dirichlet boundary conditions $\delta A|_\partial = 0$ are placed on the gauge fields. Those boundary conditions fix the potential in the thermodynamic ensemble (A_t, for example, defines the chemical potential). On the other hand, we can add boundary terms such that the action is stationary, provided the variation $\delta F|_\partial = 0$. Since the integral of F over infinity provides us with physical charges, this ensemble corresponds to the fixed-charge ensemble. The former ensemble is the grand-canonical ensemble, while the latter is the canonical ensemble. Similarly, we conjecture that the situation in the first-order formulation of gravity might be similar. We know that fixing metric using the GHY term in the second-order EH action gives the correct result for the Schwarzschild BH, and therefore, we may infer that we should add the boundary term that fixes the vielbein at the boundary. A similar conclusion was made in [78]. Although this analysis is restricted to BH thermodynamics, we can attempt to generalize its conclusions and propose that the boundary term (2.56) should always be added to the EH action in the first-order formulation. Despite correctly reproducing the GHY term for EH gravity, this motivation is not generalizable to some other gravity theory. For example, Lovelock theory in five dimensions can have the term

$$\int \varepsilon_{abcde} R^{ab} R^{cd} e^e \,, \tag{2.57}$$

whose variation produces a boundary term

$$\int \varepsilon_{abcde}\, \delta\omega^{ab} \omega^c{}_f \omega^{fd} e^e \,. \tag{2.58}$$

Unlike (2.54), we cannot move the variation from $\delta\omega^{ab}$ to δe^a, and therefore our previous analysis is not applicable here. Note that in Lovelock's theory, in five dimensions, torsion can be nonzero even on-shell, and we suspect that this might be the reason why we cannot use the same logic as in (2.54). Also note that in this paragraph, we have not been fully precise; to properly define the thermodynamical ensemble, we should more precisely define the boundary conditions (using asymptotic expansion of bulk fields).

The next attempt to better understand the role of boundary terms in first-order gravity comes from holography. We shall spend much more time discussing this in sections 3 and 4, so we only briefly comment on it here. Holographic (gauge/gravity) duality relates gravity on asymptotically AdS spacetimes with boundary field theory, and the important role is played by the set of boundary conditions. It was realized in [25] that, in order to satisfy the holographic boundary conditions (which are the Dirichlet boundary conditions for the holographic sources), appropriate finite boundary terms have to be added to the bulk action. Those boundary terms were further titled GHY type boundary terms in [79]. It is interesting to compare those boundary terms with the ones we just analysed in the case of EH theory, and we shall do this in Chapters 4 and 5. It will turn out that in a low number of dimensions, there is an agreement on the boundary terms between these two approaches.

Finally, the most modern approach to the boundary terms in the Lagrangian formulation of first-order gravity can be found in [31]. The main purpose of this paper was to set up the stage for a holographic analysis of gravity theories with torsion and non-metricity. They consider, as independent fields, frame fields, connection (the affine connection is considered in a general local basis; they do not focus on local Lorentz symmetry), and metric field, and develop a general formalism for GHY boundary terms on such spaces. The initial assumption is that what should be fixed at the boundary are the induced fields. For example, in the standard EH gravity with GHY term, one can show that it is sufficient to ask for $\delta h_{\alpha\beta}|_\partial = 0$ to make the action stationary, even though the initial motivation was to remove the variation of derivatives with respect to the boundary normal. By insisting that the similar property should be true in the case of more general gravity theories, the generalization of the GHY boundary terms can be obtained. For the purpose of this thesis, we are not interested in nonmetricity, but boundary terms for the torsionful case will be used later in the text. Note that one has to carefully define what the induced connection is at the boundary, as the connection is not a tensor field, and we cannot use a simple pullback. Let us present one example of this type of boundary terms. Five-dimensional Lovelock's theory with action

$$\int \varepsilon_{abcde}\Big(\frac{\alpha_1}{3}R^{ab}e^ce^de^e + \alpha_2 R^{ab}R^{cd}e^e + \frac{\alpha_0}{5}e^ae^be^ce^de^e\Big) \tag{2.59}$$

should be accompanied by [31, 80]

$$-2\int \varepsilon_{abcd}\Big(\frac{\alpha_1}{3}K^ae^be^ce^d + 2\alpha_2 K^aR^{bc}e^d - \frac{2\alpha_2}{3}K^aK^bK^ce^d\Big)\,, \tag{2.60}$$

where we assumed that the boundary is timelike. Here, $K^a = \mathrm{D}n^a$ is related to the boundary extrinsic curvature[2]. One can then ask whether the results of [81] regarding the entropy of the derived BH solution hold. In this paper, the authors added no boundary terms to the first-order gravity action and concluded that the on-shell action is zero, resulting in zero energy and entropy for the BH. Luckily, if we consider a black hole solution from [81] with line element

$$\mathrm{d}s^2 = -f^2(r)\mathrm{d}t^2 + \frac{1}{f^2(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\psi^2 + \sin^2\psi\mathrm{d}\theta^2 + \sin^2\psi\sin^2\theta\mathrm{d}\varphi^2)\,, \tag{2.61}$$

for $f^2(r) = \frac{\alpha_1}{8\alpha_2}\left(r^2 - \frac{r_+^8}{r^6}\right)$, and spin-connection components listed in [81], the value of the (Euclidean) on-shell action with added boundary term (2.60) is still zero

$$\frac{8\alpha_1\alpha_2 r\beta}{\alpha_2\ell^2}\int_{S^3} e^2e^3e^4 + \frac{\alpha_1^2 r\beta}{\alpha_2}\int_{S^3} e^2e^3e^4 = 0\,. \tag{2.62}$$

The last equation follows from $\alpha_1 = -8\alpha_2\frac{1}{\ell^2}$. Therefore, the results of this paper hold, which is expected because, as the authors note, the same BH thermodynamics can be derived using Nester's formula.

Finally, let us comment on one additional aspect that further supports the last claim regarding the fixing of the induced fields [82]. Despite not being fully developed, one can still

[2] Actually, the notation from [31] is slightly different from the one used in this thesis, but in the example we are interested in, boundary will be given as $r=$ const surface, such that metric tensor does not contain $g_{r\alpha}$ components; in this case, both notations coincide and we do not have to introduce any new ingreedients.

use the path integral formalism to address some aspects of quantum gravity, especially as the saddle point approximation can yield sensible results. One can compute the amplitude of a field configuration transitioning from the induced fields at one hypersurface to the fields induced at the second hypersurface. By adding the third hypersurface Σ_2 in between the first two (Σ_1 and Σ_3), we would like to compute the total transition amplitude $\langle h_3, \Sigma_3 | h_1, \Sigma_1 \rangle$. It is reasonable to expect that the following property should be true

$$\langle h_3, \Sigma_3 | h_1, \Sigma_1 \rangle = \sum_{h_2} \langle h_2, \Sigma_2 | h_1, \Sigma_1 \rangle \langle h_3, \Sigma_3 | h_2, \Sigma_2 \rangle \,, \tag{2.63}$$

where, in general, h denotes a set of induced fields at the boundary. Using the saddle point approximation, this would imply that the on-shell action computed with boundary conditions h_1 and h_3 (S_1) should be given as a sum of the on-shell action with boundary conditions h_1 and h_2 and the on-shell action with boundary conditions h_2 and h_3.

$$e^{iS} = e^{iS_1} e^{iS_2} \qquad \Rightarrow \qquad S = S_1 + S_2 \,. \tag{2.64}$$

In the case of second-order EH action, in order to fulfill this property, the GHY term has to

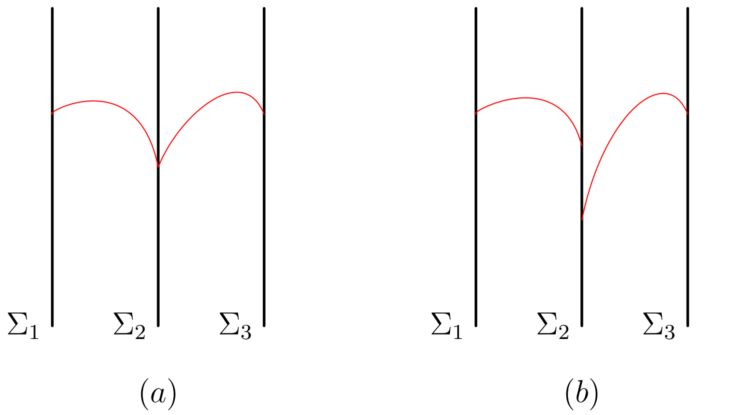

Figure 2.3: Left: in the second-order formulation of gravity, metric is continuous across Σ_2, while the derivative with respect to normal direction does not have to be; right: in the first-order formalism, connection components in arbitrary basis do not have to be continuous across Σ_2.

be added. This is because the metric field must be continuous at Σ_2, but since we only fix the induced metric, there is no reason for the derivative with respect to the boundary normal to be continuous. As standard EH action contains the second derivative of the metric field, this produces a delta function in the Lagrangian density (part (a) of Figure 2.3). On the other hand, in the first-order formulation, we fix induced vielbein and spin-connection, but unlike the metric field, there is no reason for all the components of spin-connection (in a certain basis) to be continuous. For example, in coortinate system (4.58), the components ω^{a1} do not have to be continuous across Σ_2, so that the EH Lagrangian density in (2.13), containing the first derivative of ω^{ab}, contains the delta function. If we add the boundary term (2.56), derivatives of ω are eliminated, and the total action is indeed equal to the sum of two separate parts, without the additional delta function contribution. Having discussed the boundary terms in the first-order gravity, we now turn to an important analysis of the black-hole solutions that play an enormous role in our work.

2.4 BTZ and dimensionally-continued BTZ black hole with torsion

As already pointed out, Einstein-Hilbert's gravity in three dimensions has no propagating degrees of freedom. Every solution of Einstein's equations with a negative cosmological constant is locally isomorphic to *AdS* spacetime. However, global properties of solutions can be nontrivial, and 3D gravity with a negative cosmological constant possesses a black hole solution. This black hole is called the BTZ black hole, a name stemming from the names of scientists who discovered this solution: Banados, Teitelboim, and Zanelli. Line element for this solution is given by

$$ds^2 = -\left(\frac{r^2}{\ell^2} - \mu\right) dt^2 + \frac{1}{\left(\frac{r^2}{\ell^2} - \mu\right)} dr^2 + r^2 d\theta^2. \tag{2.65}$$

Coordinate θ is angular coordinate, $\theta \sim \theta + 2\pi$. We can choose diagonal vielbeins such that

$$e^0 = \sqrt{\frac{r^2}{\ell^2} - \mu}\, dt, \quad e^1 = \frac{1}{\sqrt{\frac{r^2}{\ell^2} - \mu}}\, dr, \quad e^2 = r\, d\theta. \tag{2.66}$$

As this is a solution of Einstein's gravity, torsion vanishes, and the spin-connection is given by (analogously to (2.10))

$$\omega^{01} = \frac{r}{\ell^2} dt\,, \qquad \omega^{12} = \sqrt{\frac{r^2}{\ell^2} - \mu}\, d\theta\,. \tag{2.67}$$

The BTZ black hole has a horizon located at $r_h = \ell\sqrt{\mu}$. Using the prescription from Chapter 1, the black hole temperature is

$$T = \frac{\sqrt{\mu}}{2\pi\ell}\,. \tag{2.68}$$

Through the thesis, we will, as usual, set $\ell = 1$. Even though this black hole shares many similarities with the Schwarzschild black hole, there are important differences. At first glance, there is no problem in the metric (2.65) at $r = 0$, as $f(r) \to -\mu$ in this limit. However, the last term in the metric (2.65) is zero at $r = 0$. This is, of course, true also for the flat spacetime metric in spherical coordinates, where $r = 0$ represents a coordinate singularity. Here, however, the $r = 0$ is a true singularity. This is nontrivial to see without noting the following. As a solution to Einstein's equations in three dimensions, the BTZ black hole is locally isomorphic to *AdS* spacetime. The BTZ black hole can be obtained as a quotient spacetime from AdS, and this fact was heavily used in [83] and [84] to define a quantum version of BTZ black holes. More precisely, in [83], this observation was used to define the *Fuzzy* BTZ black hole, which is a noncommutative version of a black hole in the frame formalism (consult chapter 7), and in [84], a superposition of two BTZ black holes with different masses was constructed. One way to explicitly demonstrate this quotient structure is to start from the definition of AdS_3 space as an embedded hyperboloid from (1.34) using coordinates $(\tilde{t}, \tilde{r}, \tilde{\theta})$ as

$$V = \tilde{r}\cosh\tilde{\theta}\,, \quad U = \ell\sqrt{\frac{\tilde{r}^2}{\ell^2} - 1}\sinh\frac{\tilde{t}}{\ell}\,, \quad X_1 = \tilde{r}\sinh\tilde{\theta}\,, \quad X_2 = \ell\sqrt{\frac{\tilde{r}^2}{\ell^2} - 1}\cosh\frac{\tilde{t}}{\ell}\,. \tag{2.69}$$

These coordinates do not cover the whole AdS_3 spacetime, and they define the so-called *Rindler* AdS_3 patch. Line element of Rindler AdS_3 is given by

$$ds^2 = -\left(\frac{\tilde{r}^2}{\ell^2} - 1\right) d\tilde{t}^2 + \frac{1}{\left(\frac{\tilde{r}^2}{\ell^2} - 1\right)} d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2\,. \tag{2.70}$$

Further, we make the following rescaling of coordinates: $\tilde{t} \to t = \frac{\tilde{t}}{\sqrt{\mu}}$, $\tilde{r} \to r = \tilde{r}\sqrt{\mu}$ and $\tilde{\theta} \to \frac{\theta}{\sqrt{\mu}}$, upon which the line element takes the form of (2.65). However, a crucial point is that the Rindler AdS_3 patch is analogous to the Rindler patch in Minkowski spacetime, and even though the latter one is closely related to black holes, it is not entirely satisfactory to claim that (2.70) corresponds to a black hole (see, however, section 4.4.1). At this stage, coordinate θ is not an angular coordinate, and $\theta \in \mathbb{R}$. However, if we impose identification $\theta \sim \theta + 2\pi$, which is formally obtained by quoting the original spacetime by the action $J_n:\ \theta \to \theta + 2n\pi$, we obtain a spacetime indeed matching (2.65). Action J_n represents an infinite group $\mathbb{Z}$. This discrete identification is responsible for introducing a conical singularity at $r = 0$, and for transforming Rindler AdS_3 to a proper black hole spacetime.

We can compute the curvature two-form, using formulas analogous to (2.11), yielding

$$R^{01} = -\frac{1}{\ell^2}\mathrm{d}t\mathrm{d}r\,, \qquad R^{02} = -\frac{r}{\ell^2}\sqrt{\frac{r^2}{\ell^2} - \mu}\mathrm{d}t\ \mathrm{d}\theta\,, \qquad R^{12} = -\frac{r}{\ell^2\sqrt{\frac{r^2}{\ell^2} - \mu}}\mathrm{d}r\mathrm{d}\theta\,, \tag{2.71}$$

and it is trivial to check that $R^{ab} = -\frac{1}{\ell^2}e^a e^b$, hinting that this BH is indeed localy AdS spacetime. Furthermore, a BTZ black hole can have angular momentum J, in which case the line element is given by

$$\mathrm{d}s^2 = -\left(\frac{r^2}{\ell^2} - 8mG + \frac{16G^2J^2}{r^2}\right)\mathrm{d}t^2 + \frac{1}{\left(\frac{r^2}{\ell^2} - 8mG + \frac{16G^2j^2}{r^2}\right)}\mathrm{d}r^2 + \left(\mathrm{d}\theta + \frac{4GJ}{r^2}\mathrm{d}t\right)^2\,. \tag{2.72}$$

More precisely, J is the angular momentum assuming the bulk is described using EH action, while in the case of MB theory, the expression for the angular momentum is more complicated (see, for example, [26]).

In dimensions higher than three, we may define dimensionally-continued black holes. The line element of this spacetime is given by

$$\mathrm{d}s^2 = -\left(\frac{r^2}{\ell^2} - \mu\right)\mathrm{d}t^2 + \frac{1}{\left(\frac{r^2}{\ell^2} - \mu\right)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\psi^2 + \sin^2\psi(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\right)\,. \tag{2.73}$$

One big difference between this black hole and the original three-dimensional analogue is that here the Ricci scalar is divergent for $r \to 0$. More precisely, we have

$$R = -\frac{20}{\ell^2} + \frac{6(1+\mu)}{r^2}\,. \tag{2.74}$$

The energy of this black hole solution is given by [85]

$$M = \frac{3k\pi^2}{2\ell}\,(\mu + 2)\,\mu\,. \tag{2.75}$$

For $\mu = -1$, we obtain AdS_5 spacetime, and there is no singularity at $r = 0$. There is also a version of this black hole with a flat horizon, such that the line element is given by

$$\mathrm{d}s^2 = -\left(\frac{r^2}{\ell^2} - \mu\right)\mathrm{d}t^2 + \frac{1}{\left(\frac{r^2}{\ell^2} - \mu\right)}\mathrm{d}r^2 + r^2\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)\,. \tag{2.76}$$

In this case, the Ricci scalar is given by

$$R = -\frac{20}{\ell^2} + \frac{6\mu}{r^2}\,, \tag{2.77}$$

and now it has no singularity for $\mu = 0$. For this particular value of μ, we see that the obtained metric is that of the AdS_5 spacetime in Poincaré coordinates (1.32), once we put $\mathsf{z} = \frac{l^2}{r}$ (note a different font used for the holographic radial direction). Dimensionally-continued BTZ black hole solves the equations of motion for $5D$ AdS CS gravity. Even more interestingly, we can solve equations for CS gravity without matter fields such that the torsion is nonzero [86]. Let us choose a diagonal vielbein for the metric (2.73) and (2.76) such that

$$e^0 = \sqrt{\frac{r^2}{\ell^2} - \mu}\,\mathrm{d}t, \qquad e^1 = \frac{1}{\sqrt{\frac{r^2}{\ell^2} - \mu}}\mathrm{d}r\,, \tag{2.78}$$

and

$$e^2 = r\mathrm{d}\psi, \qquad e^3 = r\sin\psi\mathrm{d}\theta, \qquad e^4 = r\sin\psi\sin\theta\mathrm{d}\phi \tag{2.79}$$

or

$$e^2 = r\mathrm{d}x, \qquad e^3 = r\mathrm{d}y, \qquad e^4 = r\mathrm{d}z\,. \tag{2.80}$$

In both cases, the nonzero components of torsion are given by

$$T^i = -\frac{C}{r}\varepsilon^{ijk}e_je_k\,. \tag{2.81}$$

where index $i \in (2,3,4)$. A similar conclusion applies to a general horizon geometry Σ_3. We see that this torsion introduces a new constant C, and we will try to understand its role in the context of holography and black hole thermodynamics. Note that, generically, C is a new integration constant, and only for certain solutions preserving some portion of supersymmetry (half BPS solutions), this constant cannot be considered as an independent integration constant, and in those cases can be shifted to one.

For latter purposes, we will need the BTZ metric for a nonrotating black hole in Fefferman-Graham-like coordinates, recal also (1.37). An important property of this coordinate system is that the component of the metric tensor $g_{\rho\rho}$, where ρ is the radial direction in the bulk, takes the form $g_{\rho\rho} = \frac{1}{4\rho^2}$, so we make the following change of variables

$$\frac{\mathrm{d}r}{\sqrt{\frac{r^2}{\ell^2} - \mu}} = -\ell\frac{\mathrm{d}\rho}{2\rho}\,. \tag{2.82}$$

In addition, components $g_{\rho\alpha}$ should be zero, as they are in our case. Integrating (2.82), we obtain

$$\ell\ln\left(\frac{r}{\ell} + \sqrt{\frac{r^2}{\ell^2} - \mu}\right) = -\frac{\ell}{2}\ln\rho + c\,, \tag{2.83}$$

where c is an integration constant. Equivalently, we have

$$\rho = \frac{1}{\left(\frac{r}{\ell} + \sqrt{\frac{r^2}{\ell^2} - \mu}\right)^2}e^{\frac{2c}{\ell}}\,. \tag{2.84}$$

At this point, we have an ambiguity in choosing a constant c, which is nothing but the ambiguity in rescaling the radial coordinate in the FG gauge. Taking $c = 0$, we obtain

$$r = \frac{\ell}{2}\left(\frac{1}{\sqrt{\rho}} + \mu\sqrt{\rho}\right)\,, \tag{2.85}$$

which matches the choice made in [87]. If we, however, choose $c = \ell \ln 2$, we obtain a more natural (for later purposes) choice of radial coordinate, where

$$r = \ell \left(\frac{1}{\sqrt{\rho}} + \frac{\mu}{4}\sqrt{\rho} \right) . \tag{2.86}$$

Line element for the BTZ BH in these coordinates takes the form

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho}\left(- \left(1 - \frac{\mu}{4}\rho + \frac{\mu^2}{16}\rho^2\right)\mathrm{d}t^2 + \left(1 + \frac{\mu}{4}\rho + \frac{\mu^2}{16}\rho^2\right)\mathrm{d}\theta^2 \right) . \tag{2.87}$$

Chapter 3

Holography

AdS/CFT emerged from string theory as a duality between string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM. For the purpose of this thesis, the exact steps of the original derivations in [3] are not important, and we shall not spend too much time analyzing them. A very concise formulation of the derivation can be found, for example, in [88], and for convenience, we restate it here. String theory is a particular way of defining a quantum theory of gravity that starts from relativistic strings as fundamental constituents of nature and is equipped with many dualities. A particular type of nonperturbative object present in string theory is a D-brane, introduced initially as a hypersurface on which open strings can end. Starting point in [3] is a configuration of $N \gg 1$ coincident $D3$ branes (meaning that the dimension of the hypersurface is four), which backreact to deform the spacetime to

$$\mathrm{d}s^2 = \left(1 + \frac{N\tilde{l}_p}{r^4}\right)^{-\frac{1}{2}} (-\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2) + \left(1 + \frac{N\tilde{l}_p}{r^4}\right)^{\frac{1}{2}} (\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2)\,, \tag{3.1}$$

where $\tilde{l}_p$ is a constant related to the Planck's constant. Taking the limit of small r and changing variables as $z = \sqrt{N}\frac{\tilde{l}_p^2}{r}$, we obtain the line element

$$\mathrm{d}s^2 = \sqrt{N}\tilde{l}_p^2 \left(\frac{1}{z^2}\left(-\mathrm{d}x_0^2 + \mathrm{d}z^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2\right) + \mathrm{d}\Omega_5^2\right)\,, \tag{3.2}$$

where we recognize AdS_5 spacetime (1.32) and the line element of a five-sphere, so that the total manifold is $AdS_5 \times \mathbb{S}^5$. It turns out that in the limit $r \to 0$ is the so-called decoupling limit, where D branes do not interact with other ingredients in the theory, and their dynamics is described using a supersymmetric version of the Yang-Mills theory: $\mathcal{N} = 4$ super Yang-Mills in four dimensions, with gauge group $SU(N)$. Hence, there is a relation between string theory (and, thus, quantum gravity) on $AdS \times \mathbb{S}^5$ and quantum field theory (a specific one) in four dimensions. Furthermore, $\mathcal{N} = 4$ super Yang-Mills is actually conformally invariant, and therefore, we reached the AdS/CFT duality. In its full generality, this duality should relate the whole, nonperturbative theory of quantum gravity to CFT. Still, the validity of this claim is hard to prove, as both string theory and QFT are best understood as perturbative theories. Therefore, most of the research conducted in this direction works within a limit where the gravitational side of the correspondence is well-approximated using the classical theory of gravity. One should, of course, have in mind that this is just a limit, and eventually we should go beyond it. As a matter of nomenclature, we will call those considerations that derive the duality from string-theoretic arguments a *top-down* approach to holographic duality (or AdS/CFT) and those that are based on analysing certain low-energy approximations a *bottom-up* approach.

As explained in the introduction, the main focus of this thesis is the analysis of Riemann-Cartan spacetimes on the gravitational side of the duality. In this sense, we go beyond the traditional Riemannian description of classical gravity. However, in Chapter 7 we will try to formulate a boundary theory where the higher-dimensional dual does not have a geometric spacetime described using smooth manifolds. Before doing that, we need to introduce the main technical points of classical computations, which we generalize in this thesis. We will start this section by analyzing the simplest setup in which the connection between AdS spacetime and CFT emerges: a noninteracting massive scalar field in AdS. Furthermore, we will comment on how to incorporate other matter fields and, eventually, the gravitational field into the story. Next, we will analyse five-dimensional CS gravity [25] and therefore set up a formalism for dealing with Riemann-Cartan bulk in gravity theory.

3.1 Example: free scalar field on AdS spacetime

Let $\phi(t,z,\vec{x})$ denote a scalar field in AdS spacetime, described using coordinates (1.32). Klein-Gordon equation for minimally-coupled massive scalar field in AdS geometry is

$$\Box\phi - m^2\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) - m^2\phi = 0\,. \tag{3.3}$$

For concreteness, in this subsection, we will explicitly do the computations in AdS_3, but we will state the results also for any number of spacetime dimensions. Substituting Poincaré coordinates (1.32), equation (3.3) becomes

$$\Big(z^2\left(-\partial_t^2+\partial_x^2\right) + z^2\partial_z^2 - z\partial_z - m^2\Big)\phi = 0\,. \tag{3.4}$$

Because of the translational symmetry in t and x directions, we take the following ansatz, $\phi = e^{-i\omega t + ikx}f(z)$, which results in the following equation in the holographic radial coordinate

$$\Big(z^2\left(\omega^2-k^2\right) + z^2\frac{\mathrm{d}^2}{\mathrm{d}z^2} - z\frac{\mathrm{d}}{\mathrm{d}z} - m^2\Big)f(z) = 0\,, \tag{3.5}$$

with two independent solutions $z\,J_\nu(\sqrt{\omega^2-k^2}z)$ and $z\,Y_\nu(\sqrt{\omega^2-k^2}z)$. In this formulas J_ν and Y_ν are Bessel's functions of order $\nu = \sqrt{1+m^2}$. In order to quantize the theory, we shall work only with normalizable modes under the scalar product

$$(\phi_1,\phi_2) = -i\int_\Sigma \mathrm{d}\Sigma^\mu\Big(\phi_1(\partial_\mu\phi_2^*) - (\partial_\mu\phi_1)\phi_2^*\Big)\,, \tag{3.6}$$

with Σ being a Cauchy hypersurface, henceforth taken to be given by $t=\text{const}$, and therefore the only nonzero component of $\mathrm{d}\Sigma^\mu$ is

$$\mathrm{d}\Sigma^0 = \mathrm{d}z\mathrm{d}x\sqrt{|h|}n^0 = \mathrm{d}z\mathrm{d}x\frac{1}{z}\,. \tag{3.7}$$

Here, h is the induced metric on Σ, and n^μ is a normalized normal to Σ. As metric (1.32) is diagonal, it is given by $n^\mu = (A,0,0,0)$, where $A = z$ follows from $n^\mu n_\mu = -1$. Mode $z\,J_\nu(\sqrt{\omega^2-k^2}z)$, in the lmit of $z\to 0$, behaves as $\sim z^\Delta$, while mode $z\,Y_\nu(\sqrt{\omega^2-k^2}z)$ in this limit behaves as $\sim z^{2-\Delta}$. For $\Delta > 2$ (which implies $m^2 > 0$), only the first mode is normalizable, as $\int \mathrm{d}z\, \frac{1}{z} z^{2\Delta}$ converges around zero for $2\Delta - 1 > -1$. Note that for $\Delta \in [0,2]$, both modes are normalizable. Even though this regime implies $m^2 \leq 0$, it turns out that it is consistent for the m^2 parameter to be negative in AdS spacetime, as long as $\Delta \geq 0$. Scalar field is stable in this regime of parameters, and the bound $m^2 \geq -\frac{d^2}{4}$ is called the BF (Breitenlohner-Freedman) bound [89].

Finaly, we should normalize modes $\phi_{\omega,k}(t,z,x) = c\, e^{-i(\omega t - kx)}\, z\, J_\nu\left(\sqrt{\omega^2 - k^2}\, z\right)$. Direct computation gives

$$
\begin{aligned}
(\phi_{\omega_1,k_1}, \phi_{\omega_2,k_2}) &= -i|c|^2 \int_\Sigma \mathrm{d}z \mathrm{d}x\, z \Bigg(i\omega_2 e^{-i(\omega_1 t - k_1 x) + i(\omega_2 t - k_2 x)} J_\nu\left(\sqrt{\omega_1^2 - k_1^2}\, z\right) J_\nu\left(\sqrt{\omega_2^2 - k_2^2}\, z\right) \\
&\qquad + (1 \leftrightarrow 2)^* \Bigg) \\
&= 4\pi |c|^2 \delta(k_1 - k_2) \frac{\omega}{\sqrt{\omega^2 - k^2}} \delta\left(\sqrt{\omega_1^2 - k_1^2} - \sqrt{\omega_2^2 - k_2^2}\right) \\
&= 4\pi |c^2| \delta(k_1 - k_2) \delta(\omega_1 - \omega_2)\,.
\end{aligned}
\tag{3.8}
$$

We, therefore, use modes

$$
\phi_{\omega,k}(t,z,x) = \frac{z}{\sqrt{4\pi}}\, e^{-i(\omega t - kx)} J_\nu\left(\sqrt{\omega^2 - k^2}\, z\right)\,, \tag{3.9}
$$

in the rest of the computation. For future references, note that in the case of AdS_2, we would obtain modes

$$
\phi_\omega(t,z) = \frac{1}{2}\sqrt{z} J_\nu(\omega z)\,. \tag{3.10}
$$

The next step is to formulate a quantum field theory in a curved background and calculate the bulk two-point function

$$
\langle 0 | \phi(x_1, z_1) \phi(x_2, z_2) | 0 \rangle\,. \tag{3.11}
$$

We perform canonical quantization, with (3.9) for $\omega > 0$ as positive energy modes, and corresponding creation and annihilation operators $a^\dagger_{\omega,k}$ and $a_{\omega,k}$. Quantum field is expanded as

$$
\phi(z,x,t) = \int_0^\infty \mathrm{d}\omega \int_{-\omega}^{\omega} \mathrm{d}k\, \left(\phi_{\omega,k}\, a_{\omega,k} + \phi^*_{\omega,k}\, a^\dagger_{\omega,k}\right)\,, \tag{3.12}
$$

where $\left[a_{\omega_1,k_1}, a^\dagger_{\omega_2,k_2}\right] = \delta(\omega_1 - \omega_2)\delta(k_1 - k_2)$ and $a_{\omega,k}|0\rangle = 0$. We integrate only over modes with $k^2 < \omega^2$, otherwise the argument of Bessel's function becomes imaginary, and modes are no longer normalizable. A simple computation then yields the two-point function

$$
\frac{z_1 z_2}{4\pi} \int_0^\infty \mathrm{d}\omega \int_{-\omega}^{\omega} \mathrm{d}k\, e^{-i\omega(t_1 - t_2) + ik(x_1 - x_2)} J_\nu\left(\sqrt{\omega^2 - k^2}\, z_1\right) J_\nu\left(\sqrt{\omega^2 - k^2}\, z_2\right)\,. \tag{3.13}
$$

Performing the change of variables

$$
\omega = \Omega \cosh\theta\,, \qquad k = \Omega \sinh\theta\,, \tag{3.14}
$$

results in

$$
\frac{z_1 z_2}{4\pi} \int_0^\infty \mathrm{d}\Omega\, \Omega\, J_\nu(\Omega z_1) J_\nu(\Omega z_2) \int_{-\infty}^{\infty} \mathrm{d}\theta\, e^{-i\Omega\,(t_1 - t_2)\cosh\theta + i\Omega\,(x_1 - x_2)\sinh\theta}\,. \tag{3.15}
$$

The second integral may be computed using the identity [90]

$$\int_{-\infty}^{\infty} \mathrm{d}\theta \; e^{iz\cosh\theta + i\zeta\sinh\theta} = \pi i H_0^{(1)}\left(\sqrt{z^2-\zeta^2}\right) , \qquad \mathrm{Im}(z \pm \zeta) > 0 \,. \tag{3.16}$$

As we work in Lorentzian signature, we have to choose a contour of integration suitably. We take the $i\epsilon$ perscription to set $z = -\Omega t_{12} + i\epsilon$ and $\zeta = \Omega x_{12}$ in (3.16) to get

$$\frac{z_1 z_2}{2\pi} \int_0^{\infty} \mathrm{d}\Omega\, \Omega\, J_\nu(\Omega z_1)\, J_\nu(\Omega z_2)\, K_0\left(\Omega\sqrt{x_{12}^2 - t_{12}^2}\right) , \tag{3.17}$$

where we set ε to zero. This integral can be analytically expressed as [91]

$$\frac{1}{2\pi} \frac{2^\nu}{\sqrt{\zeta(\zeta+4)}} \left(\zeta + 2 + \sqrt{\zeta(\zeta+4)}\right)^{-\nu} , \tag{3.18}$$

where

$$\zeta = \frac{\eta_{\mu\nu}(x_1-x_2)^\mu (x_1-x_2)^\nu + (z_1-z_2)^2}{z_1 z_2} , \tag{3.19}$$

is the *chordal distance*, related to the geodesic distance between points in AdS. Performing similar calculations in a general number of bulk dimensions ($D = d+1$), the result for the two-point function would be

$$\frac{C_\Delta}{\zeta^\Delta}\, {}_2F_1\left(\Delta, \Delta - \frac{d-1}{2}; 2\Delta - d + 1; -\frac{4}{\zeta}\right) , \tag{3.20}$$

with normalization

$$C_\Delta = \frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}}\,\Gamma\left(\Delta - \frac{d}{2} + 1\right)} , \tag{3.21}$$

and where we defined

$$\Delta = \frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + m^2} \,. \tag{3.22}$$

We are not done yet, as our final goal is to obtain boundary correlation functions. In order to do this, we use the extrapolate dictionary of AdS/CFT [92]. We assume that the boundary operators are obtained from a suitable renormalized limit of bulk fields as

$$\mathcal{O}(t, \vec{x}) = z^{-\Delta}\phi(z, t, \vec{x}) \,. \tag{3.23}$$

More precisely, when computing the correlation functions of boundary operators, we first take the bulk correlation functions, take the limit $z \to 0$, and strip off conformal $z^{-\Delta}$ factors.

In the case of AdS_3, we have

$$G_{bb}^{\mathrm{comm}}(z_1, t_1, x_1; z_2, t_2, x_2) \to (z_1 z_2)^\Delta \, \frac{1}{2\pi\left(-(t_1-t_2)^2 + (x_1-x_2)^2\right)^\Delta} , \tag{3.24}$$

so that the two-point function of the boundary operator is given by

$$G_{\partial\partial}(t_1, x_1, t_2, x_2) = \langle \mathcal{O}(t_1, x_1)\mathcal{O}(t_2, x_2)\rangle = \frac{1}{2\pi\left(-(t_1-t_2)^2 + (x_1-x_2)^2\right)^\Delta} \,. \tag{3.25}$$

This is precisely the conformal two-point function of two primary scalar fields with conformal dimension Δ, in the Lorentzian signature. CFT is more commonly studied in Euclidean signature, and the two-point function in this case is usually written as

$$\langle \mathcal{O}(\vec{x}_1)\mathcal{O}(\vec{x}_2)\rangle = \frac{C_\Delta}{|x_1 - x_2|^{2\Delta}} \,. \tag{3.26}$$

On the other hand, we can use the GPKW dictionary to obtain the boundary two-point function. Let us illustrate this. What follows is a very common derivation, and we choose to follow [93], while making some comments along the way. Consider the Euclidean action for a massive scalar field in a curved background,

$$S_E = -\frac{1}{2}\int \mathrm{d}^3x \,\sqrt{g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right) \,. \tag{3.27}$$

After partial integration, we are left with

$$S_E = \frac{1}{2}\int \mathrm{d}^3x \,\left[\phi\,\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right) - m^2\right] - \frac{1}{2}\int \mathrm{d}^3x\,\partial_\mu\left(\sqrt{g}g^{\mu\nu}\phi\partial_\nu\phi\right) \tag{3.28}$$

where only the second term is not zero on-shell. Using the Stokes theorem, this gives a boundary term [1]

$$-\int \mathrm{d}\Sigma^\mu \; g_{\mu\nu}\phi\partial^\nu\phi \,. \tag{3.29}$$

Before analysing this boundary term further, let us comment on the KG equation in Euclidean signature and its solutions. Similarly to (3.4), for the case of Euclidean AdS_3 we have

$$z^2\left(\partial_\tau^2 + \partial_x^2\right)\phi + z^2\partial_z^2\phi - z\partial_z\phi - m^2\phi = 0\,. \tag{3.30}$$

Again, because of translational invariance, we take ansatz $\phi = e^{ik_1\tau + ik_2x}f_{\vec{k}}(z)$, which is formaly introduced via Furier transform

$$\phi(\tau, z, x) = \int \frac{\mathrm{d}^2k}{(2\pi)^2}\, e^{ik_1\tau+ik_2x}f_{\vec{k}}(z)\,. \tag{3.31}$$

Being interested in the asymptotics around the boundary $z \to 0$, we make a further ansatz $f_{\vec{k}} \sim z^\alpha$. Substituting this back into (3.31), and discarding the small term $\sim k^2z^2z^\alpha$, we are left with

$$z^\alpha\left(\alpha^2 - 2\alpha - m^2\right) = 0\,. \tag{3.32}$$

This equation has two solutions

$$\alpha_1 = 1 - \sqrt{1+m^2}\,, \qquad \alpha_2 = 1 + \sqrt{1+m^2}\,. \tag{3.33}$$

A generic solution of a second-order equation (3.31) therefore behaves as

$$f_{\vec{k}}(z) \sim z^{1-\sqrt{1+m^2}}\left(\phi_0(\vec{k}) + \ldots\right) + z^{1+\sqrt{1+m^2}}(\varphi(\vec{k}) + \ldots) \tag{3.34}$$

Similar calculation for Euclidean AdS_d would result in the following asymptotic form of the field ϕ:

$$\phi(z, x^\alpha) = z^{d-\Delta}\left(\phi_0(x^\alpha) + \ldots\right) + z^\Delta(\varphi(x^\alpha) + \ldots)\,. \tag{3.35}$$

[1] Note that the bulk term would anyway vanish in the variation, as it is proportional to the equations of motion.

We can now make a more precise statement about the GPKW dictionary (1.15). Field ϕ_0 takes the role of a source in a dual field theory. From the bulk point of view, we put Dirichlet boundary conditions on this field. Field φ, on the other hand, turns out to be related to the one-point function of a dual operator - a claim that we will derive shortly. Actually, what we stated so far is true only if $m^2 > 0$, as then $\Delta > d$ and it is clear which term in (3.35) diverges for $z \to 0$ and which goes to zero. If $m^2 \leq 0$ (recall that the BF bound makes this possible), the situation is different, and we will not review it here, as we shall not encounter such situations in this thesis.

Of course, as in Lorentzian signature, the KG equation can be solved analytically, and the general solution of (3.31) is given by

$$f_{\vec{K}}(z) = a(\vec{k}) z K_\nu \left(\sqrt{k_1^2 + k_2^2}\, z \right) + b(\vec{k}) z I_\nu \left(\sqrt{k_1^2 + k_2^2}\, z \right) , \tag{3.36}$$

so that the field $\phi(\tau, z, x)$ is given by

$$\phi(\tau, z, x) = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} e^{ik_1\tau + ik_2 x} \left(a(\vec{k}) z K_\nu \left(\sqrt{k_1^2 + k_2^2}\, z \right) + b(\vec{k}) z I_\nu \left(\sqrt{k_1^2 + k_2^2}\, z \right) \right) . \tag{3.37}$$

However, the mode proportional to the modified Bessel's function I_ν diverges as $z \to +\infty$ and therefore will not be used. The remaining mode tends to zero for $z \to +\infty$, so only the $z \to 0$ boundary contribution is relevant in (3.29). In other words, (3.29) becomes

$$\int_{\Sigma_z} \mathrm{d}\tau \mathrm{d}x\, \sqrt{-g} \phi g^{zz} \partial_z \phi \, . \tag{3.38}$$

Alternatevely, for noninteger ν, the two indepdendent solutions can be choosen as proportional to $I_\nu \left(\sqrt{k_1^2 + k_2^2}\, z \right)$ and $I_{-\nu} \left(\sqrt{k_1^2 + k_2^2}\, z \right)$, both of which diverge for $z \to +\infty$. However, for a suitable choice of coefficients $a(\vec{k})$ and $b(\vec{k})$, we can impose the regularity for $z \to +\infty$. It is important to stress that the regularity for $z \to +\infty$ imposes that the ratio $\frac{\varphi}{\phi_0}$ from (3.34) is fixed. In other words, the boundary condition for $z \to 0$ will fix the holographic source ϕ_0, while the boundary condition at $z \to +\infty$ will fix the ratio $\frac{\varphi}{\phi_0}$ and it is precisely this ratio that will be considered as independent from ϕ_0 in future functional differentiations. It is convenient to define $\Pi = \sqrt{-g} g^{zz} \partial_z \phi$ and the respective Fourier transform

$$\Pi(z, \tau, x) = \int \frac{\mathrm{d}k^2}{(2\pi)^2} e^{ik_1\tau + ik_2 x} \Pi_{\vec{k}}(z) \, , \tag{3.39}$$

such that the on-shell action can be written as

$$S_E|_{\mathrm{on-shell}} = \frac{1}{2} \int_{\Sigma_z} \frac{\mathrm{d}^2 k}{(2\pi)^2} \Pi_{-\vec{k}}(z) f_{\vec{k}}(z) \, . \tag{3.40}$$

3.1.1 Holographic renormalization

In this section, we introduce an important concept of holographic renormalization. Renormalization in QFT is a well-known procedure to obtain finite, experimentally measurable values of physical observables. Similarly, here we find that the result (3.40) is divergent, due to integration over a small z region. In order to remove this divergence, the spacetime boundary is set to $z = \varepsilon$. Ideally, we would like to send $\varepsilon \to 0$, however, at this step, this is not possible as the on-shell action takes the form

$$S_E|_{\text{on-shell}} = \frac{1}{2}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\left(\varepsilon^{-2\nu}(1-\Delta)\phi_0(-\vec{k})\phi_0(\vec{k}) + \phi_0(-\vec{k})\varphi(\vec{k})\right) . \tag{3.41}$$

In order to remove this divergence, we introduce a counterterm. We take

$$S_{ct} = -\frac{1-\Delta}{2}\int \sqrt{h}\phi^2 , \tag{3.42}$$

where h is the determinant of the induced metric. It turns out that the total action $S + S_{ct}$ is finite and given by

$$S_{\text{ren}} = \frac{1}{2}(2\Delta - 1)\int \frac{\mathrm{d}^2k}{(2\pi)^2}\phi_0(-\vec{k})\varphi(\vec{k}) . \tag{3.43}$$

Interpreting the field ϕ_0 as the holographic source, the renormalized on-shell action (3.43) takes the standard form $\int J\mathcal{O}$ for coupling of the operator with respect to the source. More formaly, we should differentiate the generating functional, here given by the bulk on-shell action, with respect to the holographic source ϕ_0:

$$\langle \mathcal{O}_{\vec{k}} \rangle = (2\pi)^d \frac{\delta S_{\text{ren}}}{\delta \phi_0(-\vec{k})} . \tag{3.44}$$

Keeping the ratio $\frac{\varphi(\vec{k})}{\phi_0(\vec{k})}$ fixed, as explained before, it is easy to show that the one-point function of the operator dual to the scalar field is given by

$$\langle \mathcal{O}_{\vec{k}} \rangle = (2\Delta - 1)\varphi(\vec{k}) . \tag{3.45}$$

We have been working in the momentum basis, but it is easy to perform the Fourier transformation to get

$$\langle \mathcal{O}(\tau, x) \rangle = (2\Delta - 1)\varphi(\tau, x) . \tag{3.46}$$

This equation demonstrates that the one-point function is indeed given by the φ field. By making another functional differentiation of (3.45) with respect to ϕ_0, again taking in mind that the ration $\frac{\varphi}{\phi_0}$ is fixed, we obtain the two-point function as

$$G(\vec{k}) = (2\Delta - 1)\frac{\varphi(\vec{k})}{\phi_0(\vec{k})} . \tag{3.47}$$

To make a concrete computation, note that we have

$$zK_\nu(kz) = (kz)^{-\nu}\left(2^{\nu-1}\Gamma(\nu)z + O\left(z^3\right)\right) + (kz)^{\nu}\left(2^{-\nu-1}\Gamma(-\nu)z + O\left(z^3\right)\right) , \tag{3.48}$$

so that we have

$$G(\vec{k}) = (2\Delta - 1)\frac{\Gamma(-\nu)}{\Gamma(\nu)}2^{-2\nu}k^{2\nu} . \tag{3.49}$$

It then remains to perform the Fourier transform of this result to the position space. Even though this is not a well-defined convergent integral, just from dimensional analysis we see that the result will behave as $|\tau^2 + x^2|^{-\Delta}$. This indeed matches the result (3.26). Computing the Fourier transformation as in [93] results in

$$\langle \mathcal{O}(\tau_1, x_1)\mathcal{O}(\tau_2, x_2) \rangle = \frac{(2\Delta - 1)\Gamma(\Delta)}{\pi\Gamma(1-\Delta)}\frac{1}{((\tau_1 - \tau_2)^2 + (x_1 - x_2)^2)^{\Delta}} . \tag{3.50}$$

Apart from the mismatch in numerical overall constants (which are usually set to one), we indeed arrived at (3.26).

The previous calculation was lengthy, but it provided us with the necessary logic for the rest of this thesis. Bulk fields provide us with operators in the boundary QFT, and there is a well-established way of obtaining the correlation functions between those operators that involves solving the bulk equations of motion and (variation) of the on-shell action. This logic is a cornerstone for the bottom-up holography approach. For example, if we consider a bulk scalar field, a bulk gauge field, and AdS black hole solution, from a holographical perspective we are describing a system that has a global $U(1)$ symmetry, scalar operator $\mathcal{O}$ that can condensate and is considered on a finite temperature (the BH temeprature from the bulk point of view). This is a minimal setup needed to describe a superfluid and lays the basics for the study of the holographic superfluid in [22] (usually also referred to as the holographic superconductor).

Even though the previous example illustrates some of the themes that will be presented in this thesis, one crucial point is missed: we have not calculated the stress-energy tensor correlation functions at the boundary. In order to do so, we have to vary the spacetime metric and thus cannot focus solely on the matter fields. Hence, our first example is very atypical as it corresponds to a boundary CFT without the stress-energy tensor. The resulting boundary theory in that case is the *generalized free field theory* (GFF), also known as mean-field theory [94, 95]. In order to obtain the correlation functions of the boundary stress-energy tensor, we follow the usual steps of holographic renormalization, keeping in mind that this quantity is dual to the bulk metric field. First, an appropriate gauge is taken for a metric field, which goes under the name of *Fefferman-Graham* gauge (recall (1.37)). It was named after mathematicians Charles Fefferman and Charles Robin Graham [51], who discovered that any vacuum solution of Einstein's equations with negative cosmological constant can be expanded near the conformal boundary $\rho = 0$ as

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho} g_{\alpha\beta}(\rho, x)\mathrm{d}x^\alpha \mathrm{d}x^\beta \,, \tag{3.51}$$

where

$$g_{\alpha\beta}(\rho, x) = g_0(x)_{\alpha\beta} + \rho g_2(x)_{\alpha\beta} + \cdots + \rho^d g_d(x)_{\alpha\beta} + \ldots \tag{3.52}$$

The (Euclidean) EH action is divergent once computed on this spacetime. This is true even in the simple case of AdS spacetime. For concreteness, we demonstrate this in five dimensions. The bulk EH action is given by

$$\int_{\rho=\varepsilon} \mathrm{d}^5x\sqrt{g}(R - 2\Lambda) \tag{3.53}$$

The equations of motion imply $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0$, so that $R = \frac{10}{3}\Lambda$. The value of the bulk on-shell action is therefore

$$\int_{\rho=\varepsilon} \mathrm{d}^5x\sqrt{g}\frac{4}{3}\Lambda \,. \tag{3.54}$$

However, the metric determinant behaves near the asymptotic boundary as $\sqrt{g} = \frac{1}{\rho^3} + \ldots$, so the integral (3.54) diverges, with the leading divergence behaving as ε^{-2}. For this reason, suitable boundary terms have to be added so that the divergences are removed. A very systematic presentation of holographic renormalization is given in [96]. However, if our focus is to extract the holographic one-point functions, we can follow the approach presented in [97], which roughly can be summarized as follows. First, we solve the classical equations of motion to obtain a solution in the form of a series in the radial variable. Next, we vary the action on-shell, and focus on terms that do not contain powers of ρ coordinate. It can be shown that infinite terms are not important, as one can always add counterterms to remove them. We shall refer to this statement in the following text as the *renormalization theorem*. After proper finite boundary terms are added to ensure the holographic boundary conditions, finite terms are in the form of $\int \langle \mathcal{O} \rangle \delta\phi_0$, so we can trivially read off the one-point functions. We shall not go into details here, as in the next section we will apply this methodology to analyse the $5D$ CS gravity.

3.2 5D CS gravity with torsion: boundary spin-current

To the best of our knowledge, the first time some gravitational bulk with torsion was analysed in the first-order formalism was in [25], where five-dimensional CS gravity was discussed. The main focus of the analysis was to obtain the FG expansion directly in the first-order formalism and to extract the one-point functions of dual quantum operators. As the role of metric field is played by vielbeins, we expect the boundary stress-energy tensor to be dual to this bulk field. On the other hand, the spin-connection is considered an independent field in the bulk, due to the presence of the torsion, and we have to understand the nature of the operator dual to this bulk field. This is a true novelty of working in the first-order formalism and dealing with CS action in five dimensions, where torsion, as shown in chapter 2, does not have to vanish on-shell. As this work presents the cornerstone for some of the analysis in this thesis, we will explicitly mention some important steps from [25]. Note that five-dimensional CS gravity was considered by a similar group of authors prior to this work in [98], but in the Riemannian sector. The details of this work were also presented and generalized in [79].

The first step of the holographic analysis in [25] was to analyse the equations of motion $\left\langle T_a F^2 \right\rangle = 0$ for the $5D$ CS action, assuming that the underlying manifold has the asymptotic structure $\mathbb{R} \times \mathcal{M}_4$. Using the index ρ for the radial direction associated with the submanifold $\mathbb{R}$ in the direct product, it was shown that, relying on the results of a canonical analysis, for a generic solution of equations of motion, identity

$$F^{AB}_{\rho\alpha} = F^{AB}_{\alpha\beta} \mathcal{N}^\beta \,, \tag{3.55}$$

holds, with F being the AdS curvature. $\mathcal{N}^\beta$ are arbitrary functions, related to the gauge transformations, and we can work in a gauge where $\mathcal{N}^\beta = 0$, thus implying $F^{AB}_{\rho\alpha}$ for a generic solution. The exact nature of those generic solutions will not be of importance here, for details we refer to [99, 100]. Further gauge choice is set by taking all the ρ components of the fields to be zero, except the $e^1 = -\frac{\mathrm{d}\rho}{2\rho}$, where we split the Lorentz indices as $(1, a)$. Furthermore, we insist that the boundary, defined at the constant ρ, is orthogonal to the bulk, and therefore $e^1_\alpha = 0$. Solving the condition $F^{AB}_{\rho\alpha} = 0$ then results in the following expansion of fields

$$\begin{aligned} e^1 &= -\frac{\mathrm{d}\rho}{2\rho} \,, & e^a &= \frac{1}{\sqrt{\rho}}(\overline{e}^a + \rho \overline{k}^a) \,, \\ \omega^{a1} &= \frac{1}{\sqrt{\rho}}(\overline{e}^a - \rho \overline{k}^a) \,, & \omega^{ab} &= \overline{\omega}^{ab} \,. \end{aligned} \tag{3.56}$$

This is the Fefferman-Graham gauge, here derived for the case of five-dimensional CS gravity[2]. Note that the FG expansion is finite, while the FG expansion for five-dimensional EH gravity is not. Actually, only in three dimensions, EH gravity has finite FG expansion, which is closely

[2]The same FG gauge can be chosen in CS gravity in an arbitrary odd number of spacetime dimensions [30].

related to the CS formulation of 3D gravity. The lower indices take values in $\{0,2,3,4\}$. Bulk metric can then be written as

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{\eta_{ab}}{\rho}\Big(\overline{e}^a_\alpha \overline{e}^b_\beta + \rho(\overline{e}^a_\alpha \overline{k}^b_\beta + \overline{e}^a_\beta \overline{k}^b_\alpha) + \rho^2 \overline{k}^a_\alpha \overline{k}^b_\beta\Big)\mathrm{d}x^\alpha \mathrm{d}x^\beta\,. \tag{3.57}$$

Boundary fields e^a and ω^{ab} are independent due to the presence of torsional degrees of freedom, and correspond to the sources in the dual field theory. Further constraints involving also the field k^a follow from the bulk equations of motion

$$\begin{aligned} &\varepsilon_{abcd}F^{ab}F^{cd} = 0\,, &&\varepsilon_{abcd}F^{bc}T^d = 0\,, \\ &\varepsilon_{abcd}F^{bc}\mathrm{D}k^d = 0\,, &&\varepsilon_{abcd}\left(F^{cd}e^e k_e + 2T^c\mathrm{D}k^d\right) = 0\,. \end{aligned} \tag{3.58}$$

where $F^{ab} = \overline{R}^{ab} + 2\overline{e}^a\overline{k}^b + 2\overline{k}^a\overline{e}^b$, and all geometrical tensors are the respective boundary quantites. These equations play a prominent role in the derivation of Ward identities in the dual theory and in the anomaly computations. The GPKW dictionary can now be schematically written as

$$\int \mathcal{D}e\mathcal{D}\omega e^{iS} \approx e^{iS_{on-shell}} = e^{iW_{CFT}[e,\omega]}\,, \tag{3.59}$$

such that the one-point functions of operators dual to e^a and ω^{ab} are obtaied by varying

$$\delta W_{CFT} = \int_{\partial\mathcal{N}} \left(\delta\overline{e}^a\tau_a + \frac{1}{2}\delta\overline{\omega}^{ab}\sigma_{ab}\right). \tag{3.60}$$

There are no issues with using the Lorentzian signature, similar to [29]. Now we come to the unfortunate point where we have to change the notation slightly. When performing bulk gravitational computations to obtain boundary observables, we will denote the bulk vielbein and spin connection with hats. Furthermore, bulk Lorentzian indices will be in uppercase, while boundary indices will be in lowercase. One then starts by varying the on-shell action. From the renormalization theorem mentioned in the last section, we can focus only on ρ-independent boundary terms, because all the divergent ones can be removed by a suitable choice of boundary counterterms. The theorem itself was actually derived after the paper [25] was written (by a similar group of authors), so the original paper contains the explicit construction of counterterms. This theorem was intensively used later in [30] when generalizing results to a higher number of spacetime dimensions.

By making the variation of the action (2.32) on-shell, the bulk term vanishes, and we are left with

$$\delta S|_{\text{on-shell}} = -2k\int_{\partial\mathcal{N}} \varepsilon_{ABCDE}\left(\hat{R}^{AB} + \frac{1}{3}\hat{e}^A\hat{e}^B\right)\hat{e}^C\delta\hat{\omega}^{DE}\,. \tag{3.61}$$

After decomposing the indices as $(1,a)$ and using the FG expansion (3.56), we obtain the variation

$$\begin{aligned} \delta S|_{\text{on-shell}} =& 4k\int_{\partial\mathcal{N}} \varepsilon_{abcd}\,\left(\left(R^{ab} + 2\,e^a k^b\right)\left(-k^c\delta e^d + e^c\delta k^d\right) + \left(De^a k^b - Dk^a e^b\right)\delta\omega^{cd}\right) \\ &+ \frac{2k}{3\rho^2}\int_{\partial\mathcal{N}} \delta\left(\varepsilon_{abcd}e^a e^b e^c e^d\right) - \frac{2\kappa}{\rho}\int_{\partial\mathcal{N}}\delta\left(\varepsilon_{abcd}\left(R^{ab} + \frac{4}{3}e^a k^b\right)k^c e^d\right)\,. \end{aligned} \tag{3.62}$$

As promised, infinite terms can be removed by adding a counterterm, and we are left with a finite contribution. However, there is an important problem in variation (3.62). While variations δe^a

and $\delta\omega^{ab}$ are consistent with (3.60), the additional variation δk^a is not. We should remove this variation by adding a finite counterterm

$$4\kappa \int \varepsilon_{abcd} \left(R^{ab} + e^a e^b\right) k^c e^d \,. \tag{3.63}$$

This finite counterterm was interpreted in [79] as the GHY boundary term, as its role is precisely to enforce the Dirichlet boundary conditions on certain fields. This boundary term was also conjectured to be related to the GHY term in [31], though the exact nature of the connection between (3.63) and (2.60) is not clear to us. The total, modified action $S + S_{\text{ct}} + S_{GHY}$ has variation

$$-8k \int_{\partial\mathcal{N}} \varepsilon_{abcd} \left(\left(R^{ab} + 2\, e^a k^b\right) k^c \delta e^d - T^a k^b \delta\omega^{cd}\right) \tag{3.64}$$

so that, in accordance with (3.60), we get

$$\tau_a = \langle \mathcal{T}_a \rangle_{\text{QFT}} = -8k\, \varepsilon_{abcd} \left(R^{bc} + 2\, e^b k^c\right) k^d \,, \tag{3.65}$$

$$\sigma_{ab} = \langle \mathcal{S}_{ab} \rangle_{\text{QFT}} = -16k\, \varepsilon_{abcd}\, T^c k^d \,. \tag{3.66}$$

Let us illustrate the usage of these results. One-point functions obtained this way can help us to derive the thermodynamic properties of a black hole, which are dual to thermal states in a boundary field theory. We shall bring the metric of (2.73) into the FG form, making the same change of coordinates (2.86) as in the case of the BTZ black hole. The resulting metric is

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho}\Big((1 + \frac{\mu}{2}\rho + \frac{\mu^2}{16}\rho^2)(\mathrm{d}\psi^2 + \sin^2\psi(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)) - (1 - \frac{\mu}{2}\rho + \frac{\mu^2}{16}\rho^2)\mathrm{d}t^2\Big) \,. \tag{3.67}$$

In this section, we shall be interested in the torsionless case, so we can read of the components in (3.56) directly from the metric. The nonzero components are given by

$$e^0 = \mathrm{d}t \,, \quad e^i = e^i_S \,, ; k^0 = -\frac{\mu}{4}\mathrm{d}t, \quad k^i = \frac{\mu}{4} e^i_S, \tag{3.68}$$

where e^i_S and ω^{ab}_S stand for the vielbein and the spin-connection on $\mathbb{S}^3$

$$e^2_S = \mathrm{d}\psi \,, \qquad e^3_S = \sin\psi \mathrm{d}\theta \,, \qquad e^4_S = \sin\psi \sin\theta \mathrm{d}\varphi \,. \tag{3.69}$$

As $\mathbb{S}^3$ is a three-dimensional manifold, we define $\omega^{ij}_S = \varepsilon^{ijk}\omega_{S\,k}$. It turns out that $\omega^i_S = e^i_S$. The energy density is given by the component T^{00} of the stress-energy tensor. This means that we have

$$\epsilon = \frac{E}{V} = \langle T_{00} \rangle = -\frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g^{00}}. \tag{3.70}$$

Note that, using the concrete form of the metric and vielbein, we see that the following set of equalities holds

$$\frac{\delta W}{\delta e^0_0} = \frac{\delta W}{\delta g^{00}} \frac{\delta g^{00}}{\delta e^0_0} = \frac{\delta W}{\delta g^{00}} \frac{\delta(e^0_0 e^{00})}{\delta e^0_0} = \frac{\delta W}{\delta g^{00}} 2e^{00} = -2\frac{\delta W}{\delta g^{00}}. \tag{3.71}$$

Therefore, we need to compute the τ_0 component of (3.65). It is given by

$$\tau_0 = k\varepsilon_{ijk} \left(1 + \frac{\mu}{2}\right) \frac{\mu}{4} e^i_S e^j_S e^k_S, \tag{3.72}$$

where we have used that for a sphere, we have $R^{ij} = e^i_S e^j_S$. From this, we obtain

$$\langle T_{00}\rangle = \frac{k}{\sin^2\psi\sin\theta}\frac{\mu}{8}(\mu+2)6\sin^2\psi\sin\theta = \frac{3}{4}k\mu(\mu+2). \tag{3.73}$$

Using AdS/CFT, this gives the mass of the corresponding black hole once we integrate over the boundary volume ($\mathbb{S}^3$ with unit radius). We have

$$M = \int_{S^3}\frac{3}{4}k\mu(\mu+2)\mathrm{d}V = \frac{3\pi^2}{2}k\mu(\mu+2). \tag{3.74}$$

This result was also derived in [98] using holography, though not relying on the first-order formalism. We can now compute the entropy of this black hole using the fact that its temperature is given by (2.68). Demanding that we have $T\mathrm{d}\mathcal{S} = \mathrm{d}M$, we obtain

$$\mathrm{d}\mathcal{S} = \frac{3\pi^3 k}{\sqrt{\mu}}\left(2\mu+2\right)\mathrm{d}\mu. \tag{3.75}$$

Integrating and demanding that for $\mu = 0$ we get zero entropy, we finally obtain

$$\mathcal{S} = 4\pi^3 k\sqrt{\mu}\left(\mu+3\right)\,. \tag{3.76}$$

This result matches the one derived in [85].

3.3 Ryu-Takayanagi formula

In this section, we will analyze one of the most important formulas that appears in the study of holographic duality [23]. As this duality relates QFT to gravity, and gravity is (classically) described using geometry, it is expected that certain field-theory quantities that are usually computed by performing operations on a Hilbert space of the quantum system are given as geometrical quantities in the bulk. One such quantity is the entanglement entropy of the spacetime region at the boundary. Consider the boundary QFT and assume that on a constant time slice, we focus on a certain region $\mathcal{A}$. For illustrative purposes, we restrict to three bulk spacetime dimensions, and therefore, our constant time slice is one-dimensional. Region $\mathcal{A}$ is taken to be a single interval, see figure 3.1.

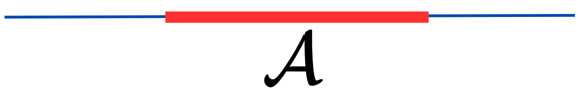

Figure 3.1: Region $\mathcal{A}$ on the line whose entanglement entropy we compute.

Assume that the system (CFT) is in a time symmetric (not necessarily pure) state ρ [88]. We can perform a partial trace of this state over all the degrees of freedom located outside of $\mathcal{A}$ to obtain a reduced density matrix

$$\rho_{\mathcal{A}} = \mathrm{Tr}_{\overline{\mathcal{A}}}\rho\,. \tag{3.77}$$

Von Neumann entropy is then defined as

$$S_{vN} = -\mathrm{Tr}\left(\rho_{\mathcal{A}}\ln\rho_{\mathcal{A}}\right)\,. \tag{3.78}$$

This entropy enjoys a few important properties. First, let us note that this quantity is well-defined for finite-dimensional quantum mechanical systems, while in the case of a field theory, it is far from trivial to see whether this quantity is well-defined. Actually, as quantum field theory generically has infinitely many degrees of freedom, it is expected that the von-Neunmann entropy will be divergent, due to a considerable amount of entanglement at the border of the region. For example, it is well-known that in CFT_2 in the ground state on a line, von-Neunmann entropy of an interval with length L is given by [101]

$$S_{EE} = \frac{c}{3}\ln\left(\frac{L}{\varepsilon}\right), \tag{3.79}$$

where ε is the UV regulator. Note that this quantity is often referred to as the entanglement entropy, because if computed for a bipartite system in a pure state, it gives a quantitative measurement of entanglement. In addition, note that to be mathematically more rigorous, one may consider relative entanglement entropy, which does not suffer from UV divergences, but we shall not analyse it here.

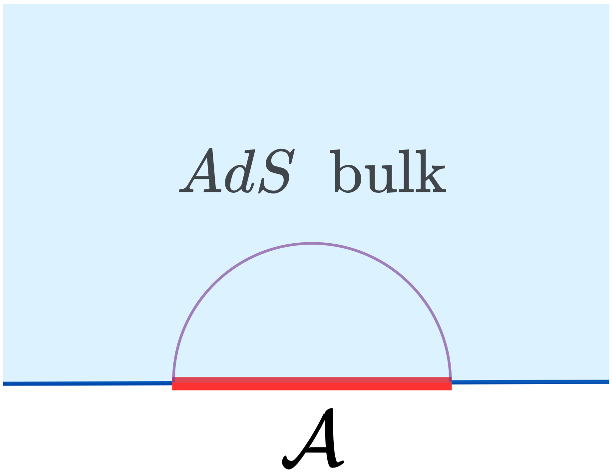

Figure 3.2: Holographic picture of computing entanglement entropy using RT formula.

It is precisely this quantity, the von-Neunmann entropy, that has a nice geometrical interpretation, using holographic duality. Assuming boundary CFT has a semi-classical bulk dual, it was conjectured by Ruy and Takayanagi [23] that the entanglement entropy is obtained as the area of a minimal codimension-two surface propagating through the bulk and anchoring the boundary at the endpoints of the respective boundary region.

$$S_{EE} = \min_{\Xi\sim L}\frac{A(\Xi)}{4G_N\hbar}. \tag{3.80}$$

For future references, we omit $\hbar$. Let us now illustrate the usage of formula (3.80) for AdS_3 in Poincaré coordinates (1.32). Geodesics anchoring the boundary $z = 0$ are given as a semicircle [102], see figure 3.2. Focusing on a boundary interval of length L, we can parametrize the geodesics as

$$z = \frac{L}{2}\sin\theta\,, \qquad x = \frac{L}{2}\cos\theta\,. \tag{3.81}$$

With this parametrization, we have

$$ds^2 = \frac{\ell^2 d\theta^2}{\sin^2\theta}\,. \tag{3.82}$$

The total length of the arc is then

$$2\int_{\frac{2\varepsilon}{L}}^{\frac{\pi}{2}} \frac{\ell \mathrm{d}\theta}{\sin\theta}\,, \tag{3.83}$$

where we had to introduce a cut-off at $z = \frac{L}{2}\sin\frac{2\varepsilon}{L} \approx \varepsilon$, for small ε. This cut-off is similar in spirit to the previously introduced cut-off in holographic renormalization. Performing this integral and using (3.80) yields the following entanglement entropy formula

$$S_{EE} = \frac{\ell}{2G}\ln\left(\frac{L}{\varepsilon}\right)\,. \tag{3.84}$$

Comparing this with (3.79) we see they both have the same form, and coincide if we take $c = \frac{3\ell}{2G}$. This is precisely the value for the boundary central charge obtained before the modern incarnation of AdS/CFT duality [15], as explained in the introduction. This simple example not only supports the validity of the RT conjecture but also illustrates an important property of AdS/CFT duality. The cut-off ε in (3.79) was introduced as an UV cut-off, present because of the divergences in QFT for large energies (small distances). On the other hand, the cut-off in (3.84) was introduced because of the infinities that arise when approaching the asymptotic infinity, implying this is an IR cut-off. The fact that (3.79) coincides with (3.84) illustrates that AdS/CFT is a UV/IR duality.

While the previous computation provides support for the validity of the RT conjecture, it does not prove it for a more general spacetime. Luckily, conjecture was proved in [103]. As far as we know, there are no generalizations of this formula to a gravity theory with Riemann-Cartan bulk. The closest approach to studying the RT formula on Riemann Cartan spacetime can be found in [62], where they used the CS description of three-dimensional gravity to reformulate the RT formula using gravitational Wilson lines. An important aspect in the derivation of the RT formula is that we need to introduce conical defects in the bulk. On the CFT side, entanglement entropy is usually derived using the replica trick, with twist operators enforcing precisely this type of singularity when moving between different sheets of the replica manifold. We shall not explain in more detail here the derivation of the RT formula, as we will try to generalize it to the Riemann-Cartan spacetime in chapter 5, relying on the CS description from [62]. In the next, final section of this Chapter, we will discuss how to incorporate boundaries into the dual field theory, leading to the AdS/BCFT duality.

3.4 AdS/BCFT

It is a mathematical fact that the boundary of a boundary of a manifold ($\partial^2\mathcal{N}$) is empty. In order to explain this, let us consider an example of a unit ball in $\mathbb{R}^3$, that is $\mathcal{B} = \{(x,y,z) \in \mathbb{R},\ x^2+y^2+z^2 \leq 1\}$. Boundary $\partial\mathcal{B}$ is given by a two-sphere $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R},\ x^2+y^2+z^2 \leq 1\}$. Clearly, S^2 is a manifold without a boundary, therefore $\partial^2\mathcal{B} = \emptyset$. Furthermore, this statement is closely related to the fact that $\mathrm{d}^2 = 0$. Integrating $\mathrm{d}^2 f$ over some n-dimenisonal manifold $\mathcal{N}$ and applying the Stokes theorem twice, we have

$$\int_{\mathcal{N}} \mathrm{d}^2 f = \int_{\partial\mathcal{N}} \mathrm{d}f = \int_{\partial^2\mathcal{N}} f. \tag{3.85}$$

Becase f is an arbitrary $n-2$ form, $\partial^2\mathcal{N} = \emptyset$. As in AdS/CFT, quantum field theory is defined on the asymptotic boundary of the bulk spacetime, and boundary theory does not have its own boundary. This contradicts many interesting examples where the boundary of real-life systems plays an important role. A CFT defined on a manifold with boundary and certain boundary conditions that preserve some conformal transformations at the boundary is called the boundary conformal field theory (BCFT) [104]. A natural question we should ask is if we can construct a bottom-up version of holographic duality, relating asymptotically AdS spacetime and boundary CFT - the AdS/BCFT duality.

A positive answer to this question was given in [105]. The boundary of a bulk spacetime is given by a union of asymptotic boundaries, where the CFT is defined, and another codimension-one hypersurface, which we will call the End-of-the-World (E.o.W.) brane (see Figure 3.3), denoted as $\mathcal{Q}$. Total action for the gravitational bulk is then

$$\frac{1}{16\pi G}\int_{\mathcal{N}} \mathrm{d}^3x\,\sqrt{-g}\,(R+2)+\frac{1}{8\pi G}\int_{\mathcal{Q}} \mathrm{d}^2y\,\sqrt{|h|}(K-T)\,, \tag{3.86}$$

(the AdS radius is set to unity) where we omitted all the boundary terms at the CFT boundary, as they should be treated in the same way as in the standard AdS/CFT. However, on the E.o.W, we added the brane tension term, as the simplest choice for the brane action (along with the necessary GHY term). Variation of the metric at the boundary then results in

$$\int_{\partial\mathcal{N}} \mathrm{d}y^2\sqrt{|h|}\,\left(K^{\alpha\beta}-(K-T)h^{\alpha\beta}\right)\delta h_{\alpha\beta}. \tag{3.87}$$

which, apart from the brane tension term, is a result following from general considerations in section 2.3.2. Here, in order to make the action stationary, we will not impose the Dirichlet boundary conditions along $\mathcal{Q}$, but rather the Neumann boundary conditions, which imply that along the brane we must have

$$K^{\alpha\beta}-(K-T)h^{\alpha\beta}=0\,. \tag{3.88}$$

This equation enables one to obtain the brane profile. A more relevant approach to our work

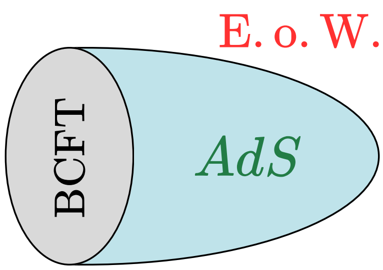

Figure 3.3: A simplified picture showing the setup for the AdS/BCFT.

was presented in [106], where the CS formulation of three-dimensional gravity (without torsion) was considered as the bulk dual in AdS/BCFT. Total action takes the form

$$\begin{aligned}&\kappa\int_{\mathcal{N}}\varepsilon_{abc}\left(R^{ab}e^c+\frac{1}{3}e^ae^be^c\right)-2\kappa\int_{\mathcal{Q}}\varepsilon_{abc}e^an^b\mathrm{d}n^c+\kappa\int_{\mathcal{Q}}e^a(\delta_a^b-n_an^b)\varepsilon_{bcd}\omega^{cd}\\&+\kappa T\int_{\mathcal{Q}}\varepsilon_{abc}n^ae^be^c\,,\end{aligned} \tag{3.89}$$

where we included the GHY term in the first-order formulation (see section 2.3.2). Again, we should impose standard Dirichlet boundary conditions (for holographic sources) at the asymptotic boundary $\mathcal{M}$ and Neumann boundary conditions at $\mathcal{N}$. Novelty in [106] is to consider also the field n^a as independent in variation. Again, the on-shell variation of the bulk part of the action vanishes, leaving us only with boundary terms. It is easy to see that the role of the GHY term, written as in (3.89), is to provide the boundary condition

$$n_a e^a|_{\mathcal{Q}} = 0\,, \tag{3.90}$$

which implies that n^a is orthogonal to the boundary Q. More precisely, if we exchange the local Lorentz index a with the coordinate one μ using vielbein, we have

$$n_a e^a_\mu \mathrm{d}x^\mu|_{\mathcal{Q}} = n_\mu \mathrm{d}x^\mu|_{\mathcal{Q}} = 0 \qquad \Rightarrow \qquad n^\mu \perp \mathcal{Q}\,. \tag{3.91}$$

Thus, the normality condition is imposed "dynamically", not postulated a priori. The other two equations are given by

$$(\eta_{ab} - n_a n_b)\,\omega^b + \epsilon_{abc} n^b \mathrm{d}n^c - T\epsilon_{abc} n^b e^c = 0\,, \tag{3.92}$$

$$(\eta_{da} - n_d n_a)\left(4\epsilon^a{}_{bc} e^b dn^c + 4n_b e^b \omega^a - T\epsilon^a{}_{bc} e^b e^c\right) = 0\,. \tag{3.93}$$

Details on the derivation of these equations, and their solutions are presented in [106]; we will not review them here as we will be dealing with the MB theory with translational CS term and without boundary gravitational anomaly in Chapter 5. Finally, note that the AdS/BCFT is closely connected to the study of the BH's information loss paradox, see for example [107].

Chapter 4

Chamseddine's topological gravity

This chapter is based on author's work [67, 108].

Chern-Simons gravity offers a well-defined gauge-theoretic formulation of gravity theory in any odd number of spacetime dimensions. The action is defined as a top form in the respective number of dimensions and does not contain the metric field. In this sense, we call this type of action topological. Note that this does not necessarily mean there is no local dynamics, and indeed, if the number of dimensions is bigger than three, local degrees of freedom exist in CS theory [99]. Unfortunately, the CS form is not defined in an even number of dimensions [109], and one has to consider a different action to generalize this topological gravity to even-dimensional space-time. The generalization was proposed in [110] by Chamseddine and will be referred to as Chamseddine's topological gravity (CTG) in this thesis. The action involves, in addition to vielbein and spin-connection, a multiplet of scalar fields. Those scalar fields are a novelty of this model and can be obtained through the process of dimensional reduction.

It is, of course, important to further motivate our study of CTG. As we will see later, in two dimensions, this model is related to JT gravity [37, 111], which has been thoroughly explored in the past. One may then say that the CTG is a natural generalization of JT gravity to any number of even dimensions. Further, CTG is used in specific approaches to four-dimensional EH gravity from the AdS gauge group [39]. Another theory, that shares similar properties, is the BF formulation of gravity in any number of spacetime dimensions. This action is often used in loop quantum gravity. However, if we are not interested precisely in Einstein's gravity, but seek some more drastic deformation of it, CTG gravity seems a more interesting candidate for our study. It is closely related to CS gravity in odd dimensions, and offers an interesting playground for understanding boundary terms in four-dimensional first-order gravity. Furthermore, as we shall see, the form of the action resembles that of a non-Abelian axion theory. Apart from motivating the CTG itself, we shall say a few words about the general idea of dimensional reduction. Through this thesis, we work in different numbers of spacetime dimensions, and one way to connect gravity theories in spacetimes with different dimensionality is precisely through this mechanism. Historically, it was realized by Kaluza and Klein [112, 113] that one can consider a five-dimensional theory of gravity to obtain an effective four-dimensional model of gravity, involving other matter fields as well. Today, the term Kaluza-Klein (KK) procedure corresponds to taking a higher-dimensional theory (usually involving gravity) and decomposing its fields in a natural way on a manifold that is a direct product of a lower-dimensional spacetime and some internal space. In a simplest example of going from a five-dimensional spacetime to a four-dimensional spacetime, the internal manifold in $\mathbb{S}^1$, though in considerations involving string theory (which is defined in ten spacetime dimensions), internal space is usually more complicated. Furthermore, fields are decomposed in terms of internal spacetime modes (for $\mathbb{S}^1$ example, this is a simple Fourier transform). Modes with higher KK momentum are heavy in the limit where the extra dimension is small and can be neglected in a low-energy approximation. Therefore, the simplest, yet meaningful approximation we can make is to consider only the subsector of zero KK modes: fields that do not depend on the internal space coordinates. We shall reserve the name *dimensional reduction*, or *KK reduction*, for this procedure. Yet, there is another motivation for considering this procedure, which does not require the existence of small extra dimensions. Often, we are interested in spacetimes with certain symmetries. For example, working with EH theory in four spacetime dimensions, we may wish to consider only spherically symmetric spacetimes. For this reason, we decompose our spacetime into $\Sigma_2 \times \mathbb{S}^2$ and assume the ansatz for our fields that respects the $SO(3)$ symmetry of a sphere: the *s-wave* approximations. This logic is more useful when starting from a realistic (or potentially interesting on its own) model and making approximations to create a simpler one, whereas the former one is more natural when trying to explain the physics around us using a higher-dimensional model.

4.1 Dimensional reduction from $5D$ CS

In this section, we will explicitly demonstrate how to obtain the $4D$ CTG starting from the $5D$ CS AdS gravity for AdS group. Most of what we say here can be repeated for the dS group, though we will not need this generalization in the thesis. This relation was proved in [110], and here we will rederive it, following the author's work in [110], see also [114]. As we shall go over different spacetime dimensions in this chapter, we will again have to introduce new notation, reserved whenever we do dimensional reduction; we believe this is, unfortunately, necessary, as any other choice would potentially induce confusion. Vielbein and spin-connection in $5D$ CS gravity will be denoted as E^A and Ω^{AB}. Five-dimensional spacetime indices will be denoted with a hat. A subsequent notation will hopefully be clear. We start with five dimenisonal AdS CS action, rewritten again as

$$S_{CS}^{(5)} = \frac{-ik}{3}\int \mathrm{Tr}\left(F^3\right) = \frac{k}{8}\int \varepsilon_{ABCDE}\Big(\frac{1}{5\ell^5}E^A E^B E^C E^D E^E + \frac{2}{3\ell^3}E^A E^B E^C R^{DE} + \frac{1}{\ell}E^A R^{BC} R^{DE}\Big) . \tag{4.1}$$

We expand spacetime indices as $\hat{\mu} = (x^\mu, y)$ where $\hat{\mu} = 0,1,2,3,4$ and $\mu = 0,1,2,3$ and $x^{\hat{4}} \equiv y$. The extra spatial dimension is compactified into a circle of radius R.

$$S_{CS}^{(5)} = -\frac{k}{8}\int \mathrm{d}^4x\mathrm{d}y\, \varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}}\varepsilon_{ABCDE}\left(\frac{1}{5\ell^5}E^A_{\hat{\mu}}E^B_{\hat{\nu}}E^C_{\hat{\rho}}E^D_{\hat{\sigma}}E^E_{\hat{\tau}} + \frac{1}{3\ell^3}E^A_{\hat{\mu}}E^B_{\hat{\nu}}E^C_{\hat{\rho}}R^{DE}_{\hat{\sigma}\hat{\tau}} + \frac{1}{4\ell}E^A_{\hat{\mu}}R^{BC}_{\hat{\nu}\hat{\rho}}R^{DE}_{\hat{\sigma}\hat{\tau}}\right) . \tag{4.2}$$

Taking only KK zero-modes into account (this allows us to integrate out y), we get

$$S_{CS}^{(5)} = -(2\pi R)\frac{k}{8}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma 4}\varepsilon_{ABCDE}\left(\frac{1}{\ell^5}E^A_{\mu}E^B_{\nu}E^C_{\rho}E^D_{\sigma}E^E_{4} + \frac{1}{\ell^3}E^A_{\mu}E^B_{\nu}E^C_{4}R^{DE}_{\rho\sigma} + \frac{2}{3\ell^3}E^A_{\mu}E^B_{\nu}E^C_{\rho}R^{DE}_{\sigma 4} + \frac{1}{4\ell}E^A_{4}R^{BC}_{\mu\nu}R^{DE}_{\rho\sigma} + \frac{1}{\ell}E^A_{\mu}R^{BC}_{\nu\rho}R^{DE}_{\sigma 4}\right) . \tag{4.3}$$

Now we unpack $A = (a, 4)$ with $a = 0, 1, 2, 3$.

$$\begin{aligned} S^{(5)}_{CS} = &-(2\pi R)\frac{k}{8}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma 4}\varepsilon_{abcd4}\Bigg(\frac{1}{\ell^5}E^a_\mu E^b_\nu E^c_\rho E^d_\sigma E^4_4 - \frac{4}{\ell^5}E^a_\mu E^b_\nu E^c_\rho E^4_\sigma E^d_4 + \frac{2}{\ell^3}E^a_\mu E^b_\nu E^c_4 R^{d4}_{\rho\sigma} \\ &+ \frac{1}{\ell^3}E^a_\mu E^b_\nu E^4_4 R^{cd}_{\rho\sigma} - \frac{2}{\ell^3}E^a_\mu E^4_\nu E^b_4 R^{cd}_{\rho\sigma} + \frac{4}{3\ell^3}E^a_\mu E^b_\nu E^c_\rho R^{d4}_{\sigma 4} + \frac{2}{\ell^3}E^a_\mu E^b_\nu E^4_\rho R^{cd}_{\sigma 4} + \frac{1}{\ell}E^a_4 R^{bc}_{\mu\nu}R^{d4}_{\rho\sigma} \\ &+ \frac{1}{4\ell}E^4_4 R^{ab}_{\mu\nu}R^{cd}_{\rho\sigma} + \frac{2}{\ell}E^a_\mu R^{bc}_{\nu\rho}R^{d4}_{\sigma 4} + \frac{2}{\ell}E^a_\mu R^{b4}_{\nu\rho}R^{cd}_{\sigma 4} + \frac{1}{\ell}E^4_\mu R^{ab}_{\nu\rho}R^{cd}_{\sigma 4}\Bigg)\,. \end{aligned} \tag{4.4}$$

To keep track of the different gauge fields presented, let us go through the details of notation regaarding the gauge fields in this section. Master gauge field $A^{\hat{A}\hat{B}}_{\hat{\mu}}$ has $SO(4,2)$ indices $\hat{A}, \hat{B} = 0, 1, 2, 3, 4, 5$ and also a spacetime index $\hat{\mu} = 0, 1, 2, 3, 4$. It is decomposed into $A^{\hat{A}\hat{B}}_{\hat{\mu}} = (A^{AB}_{\hat{\mu}} \equiv \Omega^{AB}_{\hat{\mu}}, A^{A5}_{\hat{\mu}} \equiv \frac{1}{l}E^A_{\hat{\mu}})$ with $SO(3,2)$ indices $A, B = 0, 1, 2, 3, 4$. Furthermore, we have the following structure:

$$\Omega^{AB}_{\hat{\mu}} \begin{cases} \Omega^{ab}_{\hat{\mu}} \begin{cases} \Omega^{ab}_\mu \equiv \omega^{ab}_\mu\,, \\ \Omega^{ab}_4\,, \end{cases} \\ \Omega^{a4}_{\hat{\mu}} \begin{cases} \Omega^{a4}_\mu\,, \\ \Omega^{a4}_4 \equiv -\frac{1}{\ell^2}\phi^a\,. \end{cases} \end{cases} \tag{4.5}$$

$$E^A_{\hat{\mu}} \begin{cases} E^a_{\hat{\mu}} \begin{cases} E^a_\mu \equiv e^a_\mu\,, \\ E^a_4\,, \end{cases} \\ E^4_{\hat{\mu}} \begin{cases} E^4_\mu\,, \\ E^4_4 \equiv \frac{1}{\ell}\varphi\,. \end{cases} \end{cases} \tag{4.6}$$

Note that the mass dimensions of the fields are given as $[\Omega] = 1$ and $[E] = 0$. We introduced 4-dimensional spin-connection ω^{ab}_μ and 4-dimensional vierbein e^a_μ. We also introduced scalar fields $\phi^a = \ell^2\Omega^{4a}_4$ $(a = 0, 1, 2, 3)$ and $\varphi = \ell E^4_4$, both with mass dimension -1. Multiplet $(\omega^{ab}_\mu, e^a_\mu)$ is 4d graviton; $(\Omega^{a4}_\mu, \ell^{-1}E^4_\mu)$ is 4d vector; (ϕ^a, φ) is 4d scalar multiplet.

The reduction is effected by setting $\Omega^{ab}_4 = E^a_4 = \Omega^{a4}_\mu = E^4_\mu = 0$ (truncaton of the fields) and by demanding that $\partial_4(\dots) = 0$. One should check whether this is consistent with the reduced $SO(3,2)$ gauge symmetry, and, as we shall see in the following, the answer is that it is. We are then left with a four-dimensional graviton and a scalar (in this context, usually called the dilaton or radion). From the full curvature components

$$R^{AB}_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}}\Omega^{AB}_{\hat{\nu}} - \partial_{\hat{\nu}}\Omega^{AB}_{\hat{\mu}} + \Omega^A_{\hat{\mu}C}\Omega^{CB}_{\hat{\nu}} - \Omega^A_{\hat{\nu}C}\Omega^{CB}_{\hat{\mu}}\,, \tag{4.7}$$

only $R^{ab}_{\mu\nu}$ and $R^{a4}_{\mu 4}$ components remain after reduction. In particular,

$$R^{ab}_{\mu\nu} = \partial_\mu\omega^{ab}_\nu - \partial_\nu\omega^{ab}_\mu + \omega^a_{\mu c}\omega^{cb}_\nu - \omega^a_{\nu c}\omega^{cb}_\mu\,, \tag{4.8}$$

$$R^{a4}_{\mu 4} = \partial_\mu\Omega^{a4}_4 - \partial_4\Omega^{a4}_\mu + \Omega^a_{\mu c}\Omega^{c4}_4 - \Omega^a_{4c}\Omega^{c4}_\mu = \partial_\mu\Omega^{a4}_4 + \omega^a_{\mu c}\Omega^{c4}_4 = \mathrm{D}_\mu\Omega^{a4}_4 \equiv -\frac{1}{\ell^2}\mathrm{D}_\mu\phi^a\,, \tag{4.9}$$

$$R^{ab}_{\mu 4} = \partial_\mu\Omega^{ab}_4 - \partial_4\Omega^{ab}_\mu + \Omega^a_{\mu c}\Omega^{cb}_4 - \Omega^a_{4c}\Omega^{cb}_\mu = 0\,, \tag{4.10}$$

$$R^{a4}_{\mu\nu} = \partial_\mu\Omega^{a4}_\nu - \partial_\nu\Omega^{a4}_\mu + \Omega^a_{\mu c}\Omega^{c4}_\nu - \Omega^a_{\nu c}\Omega^{c4}_\mu = 0\,. \tag{4.11}$$

Note that the covariant derivative appearing in the second equation corresponds to the four-dimensional spin-connection ω^{ab}_μ.

The reduced action becomes:

$$
\begin{aligned}
S_{reduced} &= -(2\pi R)\frac{k}{8}\int \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abcd}\Bigg(\frac{1}{\ell^5}e^a_\mu e^b_\nu e^c_\rho e^d_\sigma E^4_4 + \frac{1}{\ell^3}e^a_\mu e^b_\nu E^4_4 R^{cd}_{\rho\sigma} + \frac{1}{4\ell}E^4_4 R^{ab}_{\mu\nu}R^{cd}_{\rho\sigma} \\
&\qquad + \frac{4}{3\ell^3}e^a_\mu e^b_\nu e^c_\rho R^{d4}_{\sigma 4} + \frac{2}{\ell}e^a_\mu R^{bc}_{\nu\rho}R^{d4}_{\sigma 4}\Bigg) \\
&= -(2\pi R)\frac{k}{8\ell^3}\int \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abcd}\Bigg(\frac{1}{\ell^3}e^a_\mu e^b_\nu e^c_\rho e^d_\sigma \varphi + \frac{1}{\ell}e^a_\mu e^b_\nu R^{cd}_{\rho\sigma}\varphi + \frac{\ell}{4}R^{ab}_{\mu\nu}R^{cd}_{\rho\sigma}\varphi \\
&\qquad - \frac{4}{3\ell^2}e^a_\mu e^b_\nu e^c_\rho \mathrm{D}\phi^d - 2e^a_\mu R^{bc}_{\nu\rho}\mathrm{D}\phi^d\Bigg) \\
&= (2\pi R)\frac{k}{8\ell^2}\int \varepsilon_{abcd}\Bigg(\frac{1}{\ell^4}e^a e^b e^c e^d\varphi + \frac{2}{\ell^2}e^a e^b R^{cd}\varphi + R^{ab}R^{cd}\varphi - \frac{4}{3\ell^3}e^a e^b e^c \mathrm{D}\phi^d \\
&\qquad - \frac{4}{\ell}e^a R^{bc}\mathrm{D}\phi^d\Bigg). \qquad (4.12)
\end{aligned}
$$

In the present form, it is not clear whether this reduced action has enlarged gauge $SO(3,2)$ symmetry. We shall next prove that it has this symmetry. Gauge indices of this algebra will be denoted by $\tilde{A} = (a,5)$ with $a = 0,1,2,3$. These gauge transformations are generated by $J_{\tilde{A}\tilde{B}}$. To make the action manifestly invariant, we introduce:

$$
\Phi = \Phi^{\tilde{A}}J_{4\tilde{A}} = \Phi^a J_{4a} + \Phi^5 J_{45} = \phi^a J_{4a} + \varphi J_{45}\,, \qquad (4.13)
$$

$$
\mathcal{F} = \frac{1}{2}\mathcal{F}^{\tilde{A}\tilde{B}}J_{\tilde{A}\tilde{B}} = \frac{1}{2}F^{ab}J_{ab} + F^{a5}J_{a5}\,. \qquad (4.14)
$$

The following action is manifestly invariant under $SO(3,2)$ gauge transformations,

$$
\begin{aligned}
S_{\text{manifest}} &= a\int \mathrm{Tr}\left(\mathcal{F}\mathcal{F}\Phi\right) = \frac{a}{4}\int \mathrm{Tr}\left(J_{\tilde{A}\tilde{B}}J_{\tilde{C}\tilde{D}}J_{4\tilde{E}}\right)\mathcal{F}^{\tilde{A}\tilde{B}}\mathcal{F}^{\tilde{C}\tilde{D}}\Phi^{\tilde{E}} \\
&= -\frac{ia}{8}\int \varepsilon_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}\tilde{E}}\mathcal{F}^{\tilde{A}\tilde{B}}\mathcal{F}^{\tilde{C}\tilde{D}}\Phi^{\tilde{E}}\,. \qquad (4.15)
\end{aligned}
$$

We further show that $S_{reduced} = S_{\text{manifest}}$,

$$
\begin{aligned}
S_{\text{manifest}} &= -\frac{ia}{8}\int \varepsilon_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}\tilde{E}}\mathcal{F}^{\tilde{A}\tilde{B}}\mathcal{F}^{\tilde{C}\tilde{D}}\Phi^{\tilde{E}} \\
&= -\frac{ia}{8}\int \varepsilon_{abcd5}\left(F^{ab}F^{cd}\Phi^5 - 4F^{ab}F^{c5}\Phi^d\right) \\
&= -\frac{ia}{8}\int \varepsilon_{abcd}\left(F^{ab}F^{cd}\varphi - \frac{4}{\ell}F^{ab}\mathrm{D}e^c\phi^d\right)\,. \qquad (4.16)
\end{aligned}
$$

After partial integration, we get

$$
\begin{aligned}
S_{manifest} &= -\frac{ia}{8}\int \varepsilon_{abcd}\left(F^{ab}F^{cd}\varphi - \frac{4}{3\ell^3}e^a e^b e^c \mathrm{D}\phi^d - \frac{4}{\ell}e^a R^{bc}\mathrm{D}\phi^d\right) \qquad (4.17) \\
&= -\frac{ia}{8}\int \varepsilon_{abcd}\left(\frac{1}{\ell^4}e^a e^b e^c e^d\varphi + \frac{2}{\ell^2}e^a e^b R^{cd}\varphi + R^{ab}R^{cd}\varphi - \frac{4}{3\ell^3}e^a e^b e^c \mathrm{D}\phi^d - \frac{4}{\ell}e^a R^{bc}\mathrm{D}\phi^d\right).
\end{aligned}
$$

which is the same as $S_{reduced}$ provided $-\frac{ia}{8} = (2\pi R)\frac{k}{8\ell^2}$, i.e. $a = \frac{i2\pi Rk}{\ell^2}$. Infinitesimal $SO(3,2)$ gauge transformation are given by

$$\delta_\epsilon A = -\mathrm{D}\epsilon = -\mathrm{d}\epsilon - [A,\epsilon] \tag{4.18}$$
$$\delta_\epsilon F = -[F,\epsilon] \tag{4.19}$$

For example, the form $\mathrm{Tr}\,(F^n)$ is manifestly gauge invariant,

$$\begin{aligned}\delta_\epsilon \mathrm{Tr}\,(F^n) &= \mathrm{Tr}\left(\delta_\epsilon F F^{n-1} + \cdots + F^{n-1}\delta_\epsilon F\right)\\ &= n\mathrm{Tr}\left([\epsilon,F]F^{n-1}\right) = n\mathrm{Tr}\left(\epsilon F^n - F\epsilon F^{n-1}\right) = 0\end{aligned} \tag{4.20}$$

After the KK-reduction, $SO(3,2)$ gauge transformations with parameter $\epsilon = \frac{1}{2}\epsilon^{ab}J_{ab} + \epsilon^{a5}J_{a5}$ (we set $\epsilon^{a4} = \epsilon^{45} = 0$), the components of the A-field change as

$$\begin{aligned}
\delta_\epsilon \Omega^{ab}_\mu &= -\partial_\mu\epsilon^{ab} - \Omega^a_{\mu c}\epsilon^{cb} + \Omega^b_{\mu c}\epsilon^{ca} - \ell^{-1}(E^a_\mu\epsilon^{b5} - E^b_\mu\epsilon^{a5})\,,\\
\delta_\epsilon \Omega^{a4}_\mu &= 0\,,\\
\delta_\epsilon \Omega^{ab}_4 &= 0\,,\\
\delta_\epsilon \Omega^{a4}_4 &= \Omega^{4c}_4\epsilon_c^{\ a} + \ell^{-1}E^4_4\epsilon^{a5}\,,\\
\ell^{-1}\delta_\epsilon E^a_\mu &= -\partial_\mu\epsilon^{a5} - \Omega^a_{\mu c}\epsilon^{c5} - \ell^{-1}E^c_\mu\epsilon_c^{\ a}\,,\\
\delta_\epsilon E^4_\mu &= 0\,,\\
\delta_\epsilon E^a_4 &= 0\,,\\
\ell^{-1}\delta_\epsilon E^4_4 &= -\Omega^4_{4c}\epsilon^{c5}\,.
\end{aligned} \tag{4.21}$$

On the other hand, action S_{manifest} is manifestly invariant under $SO(3,2)$ gauge transformations, the connection being $\mathcal{A} = \frac{1}{2}\omega^{ab}_\mu J_{ab}\mathrm{d}x^\mu + \ell^{-1}e^a_\mu J_{a5}\mathrm{d}x^\mu$. Connection and Φ field change, under $SO(3,2)$ gauge transformation, as

$$\delta_\epsilon\mathcal{A} = -\mathrm{D}_{\mathcal{A}}\epsilon\,, \tag{4.22}$$
$$\delta_\epsilon\Phi = [\epsilon,\Phi]\,, \tag{4.23}$$

and it is trivial to check that this gives us exactly the same non-trivial transformation rules as above. We thus proved that five-dimensional $SO(4,2)$ gauge transformations induce four-dimensional $SO(3,2)$ gauge transformations and that the reduced action possesses this gauge symmetry. With this, we finish the analysis of the dimensional reduction and switch to the notation tailored for holographic considerations used in (3.61). Finally, the equations of motion, obtained by varying independently with respect to $\hat{e}^A$, $\hat{\omega}^{AB}$, $\hat{\phi}^A$, and $\hat{\varphi}$ are given by

$$\varepsilon_{ABCD}\big(\hat{R}^{AB} + \hat{e}^A\hat{e}^B\big)\big(\hat{R}^{CD} + \hat{e}^C\hat{e}^D\big) = 0\,, \tag{4.24}$$
$$\varepsilon_{ABCD}\hat{T}^A\big(\hat{R}^{BC} + \hat{e}^B\hat{e}^C\big) = 0\,, \tag{4.25}$$
$$\varepsilon_{ABCD}\big(\hat{R}^{BC} + \hat{e}^B\hat{e}^C\big)\big(\mathrm{D}\hat{\phi}^A - \hat{\varphi}\hat{e}^A\big) = 0\,, \tag{4.26}$$
$$\varepsilon_{ABCD}\big(2\hat{T}^B(\mathrm{D}\hat{\phi}^A - \hat{\varphi}\hat{e}^A) + (\hat{R}^{AB} + \hat{e}^A\hat{e}^B)(\mathrm{d}\hat{\varphi} - \hat{\phi}^E\hat{e}_E)\big) = 0\,. \tag{4.27}$$

The logic of this chapter is summarized in the figure 4.1.

4.1.1 Fefferman-Graham gauge in $4D$

In section 3.2, we saw that the FG gauge for the case of $5D$ CS gravity can be chosen (modulo subtleties with the gauge transformations) as (3.56). To obtain the corresponding expansion

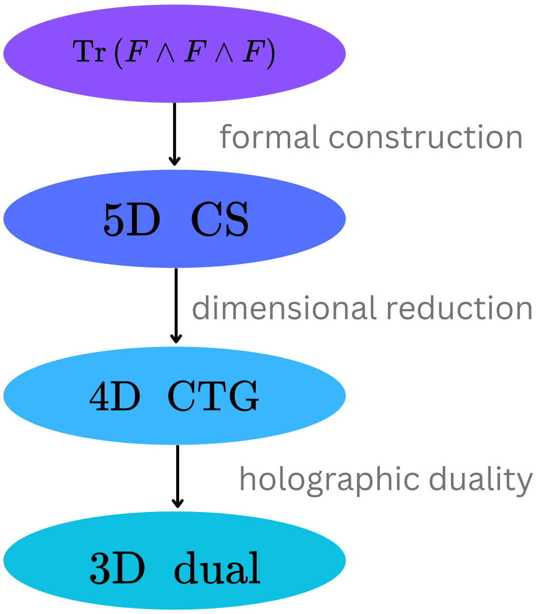

Figure 4.1: A cascade of changes in the number of spacetime dimensions used in this chapter. In the first step, we have a six-dimensional topological term $\mathrm{Tr}(F^3)$, which is used to formally construct the action for five-dimensional CS theory. We focus on CS gravity and therefore take the $SO(4,2)$ gauge group. This theory is then dimensionally reduced to four dimensions (together with performing a consistent truncation) to obtain a four-dimensional gravity theory we call Chamseddine's topological gravity (CTG). We are mostly interested in a holographic dual to this theory, which lives in three spacetime dimensions.

in CTG, we again note the following. First, the reduction procedure from CS to CTG respects symmetry as much as possible. Furthermore, the equations of motion for CTG can be obtained by reducing the ones in CS gravity. Because of these two facts, we obtain the FG gauge in four-dimensional CFT simply by reducing the one from $5D$ CS gravity. If this seems unsatisfactory, note that one can follow the same line of reasoning as in [25] to derive the corresponding formulas. We explicitly show the reduction. Vielbein and spin-connection in $4D$ are simply obtained from the five-dimensional ones as

$$\hat{e}^1 = -\frac{\mathrm{d}\rho}{2\rho}\,, \quad \hat{e}^a = \frac{1}{\sqrt{\rho}}(e^a + \rho k^a)\,, \tag{4.28}$$

$$\hat{\omega}^{a1} = \frac{1}{\sqrt{\rho}}(e^a - \rho k^a)\,, \quad \hat{\omega}^{ab} = \omega^{ab}\,. \tag{4.29}$$

A novelty of this theory is scalar fields, originating from the higher-dimensional vielbein and spin-connection. From the form of FG gauge for components $\hat{e}^5$ and $\hat{\omega}^{a5}$ we obtain the following expansion

$$\hat{\phi}^1 = \frac{1}{\sqrt{\rho}}(\varphi - \rho\psi)\,, \quad \hat{\phi}^a = \phi^a\,, \quad \hat{\varphi} = \frac{1}{\sqrt{\rho}}(\varphi + \rho\psi)\,. \tag{4.30}$$

Alternatively, one can show that, similar to (3.55), we have [115]

$$\begin{pmatrix} \hat{F}^{\hat{A}\hat{B}}_{i\rho} \\ \mathrm{D}_\rho \hat{\Phi}^{\hat{C}} \end{pmatrix} = \mathcal{N}^j \begin{pmatrix} \hat{F}^{\hat{A}\hat{B}}_{ji} \\ \mathrm{D}_j \hat{\Phi}^{\hat{C}} \end{pmatrix}\,, \tag{4.31}$$

and use the same argument as before to derive the FG gauge (4.30). In FG gauge, equations of motion (4.24)-(4.27) reduce to

$$\varepsilon_{abc}\big(\mathrm{D}\phi^a - 2k^a\varphi - 2e^a\psi\big)\big(R^{bc} + 4e^bk^c\big) = 0\,, \tag{4.32}$$

$$\varepsilon_{abc}\big[(\mathrm{d}\varphi - e_d\phi^d)\mathrm{D}k^c - (\mathrm{d}\psi - k_d\phi^d)T^c + (\mathrm{D}\phi^c - 2e^c\psi - 2k^c\varphi)e^dk_d\big] = 0\,, \tag{4.33}$$

$$\varepsilon_{abc}\big[(\mathrm{d}\varphi - e_d\phi^d)(R^{bc} + 4e^bk^c) + 2(\mathrm{D}\phi^b - 2k^b\varphi - 2e^b\psi)T^c\big] = 0\,, \tag{4.34}$$

$$\varepsilon_{abc}\big[(\mathrm{d}\psi - k_d\phi^d)(R^{bc} + 4e^bk^c) + 2(\mathrm{D}\phi^b - 2k^b\varphi - 2e^b\psi)\mathrm{D}k^c\big] = 0\,. \tag{4.35}$$

Equations (4.24) and (4.25) are automatically satisfied. We illustrate how (4.32) is derived from (4.26). By taking the last index $D = 1$ in (4.26), we have

$$\begin{aligned}
\varepsilon_{abc1}\big(\hat{R}^{bc} + \hat{e}^b\hat{e}^b\big)\big(\mathrm{D}\hat{\phi}^a - \hat{\varphi}\hat{e}^a\big) &= -\varepsilon_{abc}\left(\mathrm{d}\omega^{bc} + \omega^b{}_d\omega^{dc} + \hat{\omega}^b{}_1\hat{\omega}^{1c} + \hat{e}^b\hat{e}^b\right) \times \\
&\qquad \left(\mathrm{d}\phi^a + \omega^a{}_b\phi^b + \hat{\omega}^a{}_1\hat{\phi}^1 - \hat{\varphi}\hat{e}^a\right) \\
= -\varepsilon_{abc}\Bigg(R^{bc} - \frac{1}{\rho}\left(e^b - \rho k^b\right)\left(e^c - \rho k^c\right) &+ \frac{1}{\rho}\left(e^b + \rho k^b\right)\left(e^c + \rho k^c\right)\Bigg) \times \\
&\qquad \Bigg(\mathrm{D}\phi^a + \frac{1}{\rho}\left(e^a - \rho k^a\right)\left(\varphi - \rho\psi\right) - \frac{1}{\rho}\left(\varphi + \rho\psi\right)\left(e^a + \rho k^a\right)\Bigg) \\
= -\varepsilon_{abc}\big(\mathrm{D}\phi^a - 2k^a\varphi - 2e^a\psi\big)\big(R^{bc} + 4e^bk^c\big)\,.
\end{aligned} \tag{4.36}$$

Other equations are derived in a similar fashion.

4.1.2 Formal connection with MMCSW gravity

CTG action is not equivalent to EH gravity. However, a close inspection shows that it contains the EH term, multiplied by a scalar φ. We can then consider a symmetry breaking from $SO(3,2)$ to $SO(3,1)$ implied by taking the scalar multiplied to be of the form

$$\hat{\phi}^A = (0,0,0,0,\ell)\,. \tag{4.37}$$

The resulting action is given by

$$S_{reduced,\mathrm{SB}} = (2\pi R)\frac{k}{8\ell^3}\int \varepsilon_{ABCD}\left(\frac{1}{\ell^2}\hat{e}^A\hat{e}^B\hat{e}^C\hat{e}^D + 2\hat{e}^A\hat{e}^B\hat{R}^{CD} + \ell^2\hat{R}^{AB}\hat{R}^{CD}\right), \tag{4.38}$$

or in the second-order formalism:

$$S_{reduced,\mathrm{SB}} = (2\pi R)\frac{k}{8\ell^3}\int \mathrm{d}^4x\sqrt{-\hat{g}}\left[\frac{24}{\ell^2} + 4\hat{R} + l^2\left(\hat{R}^2 - 4\hat{R}^{\mu\nu}\hat{R}_{\nu\mu} + \hat{R}^{\mu\nu\rho\sigma}\hat{R}_{\rho\sigma\mu\nu}\right)\right]. \tag{4.39}$$

Action obtained in this fashion consists of the EH term, the cosmological constant term, and the Gauss-Bonnet term in four dimensions. However, in four dimensions, the Gauss-Bonnet term is topological. Varying

$$S_{GB} = k\int \varepsilon_{ABCD}\hat{R}^{AB}\hat{R}^{CD} \tag{4.40}$$

with respect to vielbein is trivial, and variation with respect to spin-connection gives a boundary term

$$\begin{aligned}
\delta\left(\int \varepsilon_{ABCD}\hat{R}^{AB}\hat{R}^{CD}\right) = 2\int \varepsilon_{ABCD}\hat{R}^{AB}\mathrm{D}\delta\hat{\omega}^{CD} &= 2\int \mathrm{d}\left(\varepsilon_{ABCD}\hat{R}^{AB}\delta\hat{\omega}^{CD}\right) \\
&= \int_\partial \varepsilon_{ABCD}\hat{R}^{AB}\delta\hat{\omega}^{CD}\,.
\end{aligned} \tag{4.41}$$

Similar consideration can be done in the second-order formalism. Therefore, the Gauss-Bonnet term does not change the equations of motion, and the resulting theory is Einstein's gravity with a negative cosmological constant. This procedure can formally be considered as a version of the MacDowell-Mansouri-Chamseddine-Stelle-West approach to four-dimensional AdS gravity [38, 39]. The starting idea of this formalism is to make a connection between EH gravity and gauge theory for the AdS_4 gauge group. As EH theory is invariant only under local Lorentz transformations, and not under the full $SO(3,2)$ transformations, it is necessary to impose a symmetry breaking. In this subsection, the symmetry breaking was introduced by hand, whereas one should hope that, in a more systematic treatment, it would be spontaneous symmetry breaking. Regardless of this, in what follows, we will often write

$$4\kappa = \frac{1}{16\pi G_N}\,, \tag{4.42}$$

where $\kappa = \frac{-ia}{8}$ is the constant multiplying the integral in (4.15), even though we are not dealing with the EH theory. In the next section, we compute the one-point functions of boundary operators.

4.2 One-point functions and Weyl anomaly

In order to extract the one-point function of operators dual to appropriate bulk fields, we have to compute the variation of the on-shell action with respect to boundary sources. Starting from action (4.15), it is obvious that the total value of the bulk on-shell action is zero. However, we have to take care of boundary conditions. We interpret fields e^a, ω^{ab}, φ, and ϕ^a as boundary sources and put Dirichlet boundary conditions on those fields. On the other hand, other bulk fields should be left free to vary at the boundary. We, therefore, supplement the bulk action with appropriate boundary terms such that these boundary conditions are enough to ensure the stationarity of the action. To start, note that we have

$$\kappa\int_{\mathcal{M}_4}\varepsilon_{\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}}\Big(\delta\hat{\Phi}^{\hat{A}}\hat{F}^{\hat{B}\hat{C}}\hat{F}^{\hat{D}\hat{E}} + 2\hat{\Phi}^{\hat{A}}\delta\hat{F}^{\hat{B}\hat{C}}\hat{F}^{\hat{D}\hat{E}}\Big)\,. \tag{4.43}$$

Decomposing indices, we get

$$\kappa\int_{\mathcal{M}_4}\varepsilon_{ABCD}\Big(\delta\hat{\varphi}\hat{F}^{BC}\hat{F}^{DE} + 2\delta\hat{\phi}^A\hat{T}^B\hat{F}^{CD} + 2\hat{\varphi}\delta\hat{F}^{AB}\hat{F}^{CD} + 4\hat{\phi}^A\delta\hat{T}^B\hat{F}^{CD} + 4\hat{\phi}^A\hat{T}^B\delta\hat{F}^{CD}\Big)\,.$$

After some partial integration, putting the variation on-shell yields

$$\delta S|_{\text{on-shell}} = \kappa\int_{\partial\mathcal{M}_4}\varepsilon_{ABCD}\Big(2\hat{\varphi}\delta\hat{\omega}^{AB}\hat{F}^{CD} + 4\hat{\phi}^A\delta\hat{e}^B\hat{F}^{CD} + 4\hat{\phi}^A\hat{T}^B\delta\hat{\omega}^{CD}\Big)\,. \tag{4.44}$$

Next, we will insert the FG expansion from the previous section into the variation (4.44), resulting in

$$\begin{aligned}
4\kappa\int_{\partial\mathcal{M}_4}\varepsilon_{abc}\Big(&\frac{1}{\rho}(\varphi+\rho\psi)(\delta e^a-\rho\delta k^a)(R^{bc}+4e^bk^c)+\frac{1}{\rho}(\varphi+\rho\psi)\delta\omega^{ab}(T^c-\rho\mathrm{D}k^c)\\
&-\frac{1}{\rho}(\varphi-\rho\psi)(\delta e^a+\rho k^a)(R^{bc}+4e^bk^c)+\frac{2}{\rho}\phi^a(\delta e^b+\rho\delta k^b)(T^c-\rho\mathrm{D}k^c))\\
&-\frac{1}{\rho}(\varphi-\rho\psi)(T^a+\rho\mathrm{D}k^a)\delta\omega^{bc}+\phi^a(-2e^dk_d)\delta\omega^{bc}+\frac{2}{\rho}\phi^a(T^b+\rho\mathrm{D}k^b)(\delta e^c-\rho\delta k^c)\Big)\,.
\end{aligned} \tag{4.45}$$

Normally, at this stage, we should regularize the integral by putting the boundary at $\rho = \varepsilon$ and adding counterterms so that all the infinities are removed. While this is necessary for the EH action (as demonstrated in Chapter 3), here the bulk action is zero on-shell, and we face a puzzling situation. Luckily, it turns out that there are no infinite terms in (4.45). Focusing on a term proportional to the negative power of ρ, we have

$$4\kappa \int_{\partial\mathcal{M}_4} \varepsilon_{abc}\frac{1}{\rho}\Big(\varphi\delta e^a(R^{bc}+4e^bk^c)+\varphi\delta\omega^{ab}T^c-\varphi\delta e^a(R^{bc}+4e^bk^c)+2\phi^a\delta e^bT^c \\ -\varphi T^a\delta\omega^{bc}+2\phi^aT^b\delta e^c\Big)=0\,. \tag{4.46}$$

Thus, there are no infinities to be eliminated, contrary to the example from EH gravity. Mathematically, this can be understood as a consequence of zero on-shell action supplemented further by the renormalization theorem. Next, we focus on terms that are finite in the $\rho \to 0$ limit, because, as we saw in section 3.1, they will provide us with the one-point functions of dual operators in the boundary QFT, once proper boundary conditions are imposed. Those terms give

$$\delta S|_{\text{on-shell}} = 4\kappa\int_{\partial\mathcal{M}_4}\varepsilon_{abc}\Big(\delta k^a\big(-2\varphi(R^{bc}+4e^bk^c)-4\phi^bT^c\big)+\delta e^a\big(2\psi(R^{bc}+4e^bk^c)+4\phi^b\mathrm{D}k^c\big) \\ +\delta\omega^{ab}\big(-2\varphi\mathrm{D}k^c+2\psi T^c-2\phi^ce^dk_d\big)\Big). \tag{4.47}$$

At this stage, the role of boundary conditions becomes crucial. Insisting only on Dirichlet boundary conditions for holographic sources, this variation is not stationary. This is obvious because of the term proportional to δk^a. We therefore, following the reasoning of [25] and [26] (see also section 3.2), add appropriate boundary terms, such that the variation is zero if the boundary conditions are satisfied. For example, if we focus on the first term in (4.47) proportional to δk^a, we have

$$\int_{\partial\mathcal{M}_4}\varepsilon_{abc}\delta k^a\varphi R^{bc} = \int_{\partial\mathcal{M}_4}\varepsilon_{abc}\big(\delta(k^a\varphi R^{bc})-\delta\varphi k^aR^{bc}-\mathrm{D}(k^a\varphi)\delta\omega^{bc}\big)\,, \tag{4.48}$$

where we also performed partial integration and removed boundary term $\mathrm{d}(\varepsilon_{abc}\varphi k^a\omega^{bc})$, because $\partial^2\mathcal{M} = \emptyset$. Performing similar calculations for the rest of the terms, we conclude that by adding the boundary term

$$S_{GHY} = 8\kappa\int_{\partial\mathcal{M}_4}\varepsilon_{abc}\left(\varphi k^aR^{bc}+2\varphi k^ak^be^c+2k^a\phi^bT^c\right) \tag{4.49}$$

to the action (4.15), resulting variation $S_{\text{mod}} = S_{\text{manifest}} + S_{GHY}$ does not contain δk^a, and thus can be given a holographic interpretation. Relying on Chapter 3, we can write

$$\delta S_{\text{mod}} = \delta W = \int_{\partial\mathcal{M}_4}\left(\delta e^a\tau_a+\frac{1}{2}\delta\omega^{ab}\sigma_{ab}+\delta\varphi o_\varphi+\delta\phi^ao_a\right), \tag{4.50}$$

where, in addition to τ_a and σ_{ab} from section 3.2, we introduced charges o_φ and o_a, which provide us with boundary operators dual to the dilaton-like bulk scalar fields. The explicit computation gives

$$\delta S_{\text{mod}} = \frac{1}{16\pi G_N}\int_{\partial\mathcal{M}_4}\varepsilon_{abc}\Big(\delta e^a\big(2\psi(R^{bc}+4e^bk^c)-4k^b\mathrm{D}\phi^c+4k^bk^c\varphi\big) \\ +\delta\varphi\big(2k^aR^{bc}+4k^ak^be^c\big)+\delta\phi^a\big(-4k^bT^c\big)\Big)+\delta\omega^{ab}\big(\varepsilon_{abc}(2\psi T^c-2\phi^ce^dk_d-2k^c\mathrm{d}\varphi) \\ -4\varepsilon_{acd}k^c\phi^de_b\big)\Big)\,. \tag{4.51}$$

such that we can read off the one-point functions as

$$
\begin{aligned}
\tau_a = \langle \mathcal{T}_a \rangle_{\mathrm{QFT}} &= \frac{1}{16\pi G_N}\varepsilon_{abc}\big(2\psi(R^{bc}+4e^b k^c) - 4k^b \mathrm{D}\phi^c + 4k^b k^c \varphi\big)\,,\\
\sigma_{ab} = \langle \mathcal{S}_{ab} \rangle_{\mathrm{QFT}} &= \frac{1}{16\pi G_N}\varepsilon_{abc}(2\psi T^c - 2\phi^c e^d k_d - 2k^c \mathrm{d}\varphi) - 4\varepsilon_{acd} k^c \phi^d e_b\,,\\
o_\varphi = \langle \mathcal{O}_\varphi \rangle_{\mathrm{QFT}} &= \frac{1}{8\pi G_N}\varepsilon_{abc}(k^a R^{bc} + 2k^a k^b e^c)\,,\\
o_a = \langle \mathcal{O}_a \rangle_{\mathrm{QFT}} &= -\frac{1}{4\pi G_N}\varepsilon_{abc} k^b T^c\,. \qquad (4.52)
\end{aligned}
$$

These expressions constitute one of the main results of this chapter. Once we have obtained them, we can continue to discuss the Weyl anomaly.

It is a well-known fact that conformal invariance implies that, classically, the trace of the energy-momentum tensor vanishes [116]. To prove this, we consider the theory defined by the action $\int \mathrm{d}^d x \sqrt{|g|}\mathcal{L}$ and Weyl rescalings of a metric $g_{\mu\nu} \to e^{\omega(x)} g_{\mu\nu}$ that preserve the action. Infinitesimal transformation is then given by $\delta g_{\mu\nu} = \omega(x) g_{\mu\nu}(x)$. We have

$$\delta S = \int \mathrm{d}^d x\, \frac{\delta\left(\sqrt{-g}L\right)}{\delta g_{\mu\nu}} \delta g_{\mu\nu} = -2\int \mathrm{d}x^d\, \sqrt{-g} T^{\mu\nu}\omega(x) g_{\mu\nu}\,. \qquad (4.53)$$

On the other hand, if Weyl rescalings are symmetries of the theory (and they are, if we are interested in CFTs), we have $\delta S = 0$, which in turn implies that the stress-energy tensor is traceless: $T^\mu{}_\mu = 0$.

However, once quantum effects are taken into account, the expectation value $\langle T^\mu{}_\mu \rangle$ does not have to vanish. For example, for a two-dimensional CFT, this expectation value is proportional to the Ricci curvature of the underlying Riemann manifold, where the proportionality constant is given by the central charge of this CFT. This is an example of a quantum anomaly called the Weyl anomaly. Contrary to this example, in any odd number of spacetime dimensions, there is no Weyl anomaly. As CTG, as we discussed in this chapter, is defined for even D, the boundary of the spacetime has an odd number of dimensions d, and therefore, we should get that the Weyl anomaly is not present. Let us show that this is indeed the case. We have to be precise about which expression we have to compute, as the nontrivial profile of the dilaton-like fields generically leads to sources in the dual theory, and therefore, we do not expect that the stress-energy tensor is traceless even without the anomaly. From a different perspective, field $\hat{\varphi}$ comes from the vielbein component $\hat{e}_5^5$, and the five dimensional anomaly expression $e^A \langle \mathcal{T}_A \rangle_{\mathrm{QFT}}$ contains the term $\varphi \langle \mathcal{O}_\varphi \rangle_{\mathrm{QFT}}$. The expression we have to compute is therefore

$$\mathcal{A} \equiv e^a \langle \mathcal{T}_a \rangle_{\mathrm{QFT}} + \varphi \langle \mathcal{O}_\varphi \rangle_{\mathrm{QFT}}\,. \qquad (4.54)$$

The explicit computation, using (4.26), gives

$$
\begin{aligned}
\mathcal{A} &= \frac{1}{16\pi G_N}\varepsilon_{abc}\left(2\psi e^a R^{bc} + 8\psi e^a e^b k^c - 4e^a k^b \mathrm{D}\phi^c + 8\varphi e^a k^b k^c + 2\varphi k^a R^{bc}\right)\\
&= \frac{1}{16\pi G_N}\varepsilon_{abc}\left(\mathrm{D}\phi^a R^{bc} - \left(\mathrm{D}\phi^a - 2k^a\varphi - 2e^a\psi\right)\left(R^{bc} + 4e^b k^c\right)\right)\\
&= \frac{1}{16\pi G_N}\varepsilon_{abc}\mathrm{D}\phi^a R^{bc} = \mathrm{d}\left(\frac{1}{16\pi G_N}\varepsilon_{abc}\phi^a R^{bc}\right) = 0\,, \qquad (4.55)
\end{aligned}
$$

where in the last step we have neglected the total derivative because it can be removed by a suitable redefinition, as standard in anomaly computations. Thus, the Weyl anomaly indeed vanishes in three-dimensional boundary theory.

4.2.1 Classical bulk solutions to CTG

The set of equations (4.24)-(4.27) has to be solved in order to obtain bulk geometries. As usual in gravity theories, this is a set of coupled partial differential equations, and their solutions are generically hard to obtain. As 4D CFT was obtained by dimensional reduction from 5D CS gravity, it is natural to seek a black hole solution that is a reduction of the black hole from (2.73) and (2.76). For concreteness and possible applications in AdS/CMT, we will develop the case with the flat horizon.

The line element of the spacetime is given by

$$\mathrm{d}s^2 = -\left(r^2-\mu\right)\mathrm{d}t^2 + \frac{1}{\left(r^2-\mu\right)}\mathrm{d}r^2 + r^2\left(\mathrm{d}x^2+\mathrm{d}y^2\right)\ . \tag{4.56}$$

and the scalar fields are given by

$$\hat{\varphi} = r\,,\quad \hat{\phi}^1 = \sqrt{r^2-\mu}\,,\quad \hat{\phi}^a = 0\,. \tag{4.57}$$

Unlike in 5D CS gravity, torsion vanishes for this solution. In the FG gauge, metric takes the form

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho}\Big((1+\frac{\mu}{2}\rho+\frac{\mu^2}{16}\rho^2)(\mathrm{d}x^2+\mathrm{d}y^2) - (1-\frac{\mu}{2}\rho+\frac{\mu^2}{16}\rho^2)\mathrm{d}t^2\Big). \tag{4.58}$$

Boundary fields are given by

$$e^a = \delta^a_\mu \mathrm{d}x^\mu\,,\,\quad \omega^{ab} = 0\,, \tag{4.59}$$

$$k^a = \epsilon\frac{\mu}{4}\delta^a_\mu \mathrm{d}x^\mu\,,\quad \varphi = 1\,, \tag{4.60}$$

$$\psi = \frac{\mu}{4}\,,\quad \phi^a = 0\,. \tag{4.61}$$

It is straightforward to check that (4.59)-(4.61) satisfy (4.32)-(4.35). Furthermore, we can compute the one-point functions of the stress-energy tensor. They are given by

$$\langle \mathcal{T}_0\rangle_{\mathrm{QFT}} = \frac{3\mu^2}{32\pi G}\mathrm{d}x^2\mathrm{d}x^3\,, \tag{4.62}$$

$$\langle \mathcal{T}_2\rangle_{\mathrm{QFT}} = \frac{\mu^2}{32\pi G}\mathrm{d}x^0\mathrm{d}x^3\,, \tag{4.63}$$

$$\langle \mathcal{T}_3\rangle_{\mathrm{QFT}} = -\frac{\mu^2}{32\pi G}\mathrm{d}x^0\mathrm{d}x^2\,, \tag{4.64}$$

and the one-point functions for operators $\mathcal{O}_\varphi$ and $\mathcal{O}_a$ are

$$\langle \mathcal{O}_\varphi\rangle_{\mathrm{QFT}} = -\frac{\mu^2}{32\pi G}\mathrm{d}x^0\mathrm{d}x^2\mathrm{d}x^3\,, \tag{4.65}$$

$$\langle \mathcal{O}_a\rangle_{\mathrm{QFT}} = 0\,. \tag{4.66}$$

It is trivial to check that the condition of vanishing Weyl anomaly (4.54) is fulfilled.

Further solutions to four-dimensional CTG were recently presented in [57]. A positive aspect of the analysis done in this paper is the finding of new solutions with torsion. The focus in [57] was on both AdS and dS gravity, but for the purpose of this thesis, we will discuss only AdS gravity. For example, it was shown that the spacetime with the line element of the form

$$\mathrm{d}s^2 = -\left(1 + r^2 \pm \sqrt{a + br}\right)\mathrm{d}t^2 + \frac{1}{\left(1 + r^2 \pm \sqrt{a + br}\right)}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2 \tag{4.67}$$

and zero torsion solves equations of motion, provided the scalar fields are zero. Depending on the choice of $\pm$ sign and parameters a and b, this solution may represent a black hole with one or two horizons, a naked singularity, or, for a particular case $a = b = 0$, a global AdS spacetime, which we already discussed. Apart from the trivial case, for nonzero a and b, this spacetime metric cannot be put in the FG form discussed above. This is not unexpected, as the solution has many field components that are set to zero, and our derivation was restricted to the case of generic solutions. However, this is not a problem, because we can view our FG expansion as an asymptotic expansion, where all the steps are essentially the same as those done. Another solution is given by the line element

$$\mathrm{d}s^2 = -\left(r^2\left(1 \pm \frac{r^2}{r_h^2}\right)\right)\mathrm{d}t^2 + \frac{1}{\left(r^2\left(1 \pm \frac{r^2}{r_h^2}\right)\right)}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\,, \tag{4.68}$$

and the nonzero components of the torsion two-form are given by

$$T^0 = -2\sqrt{1 \pm \frac{r^2}{r_h^2}}\,\mathrm{d}t\mathrm{d}r\,, \qquad T^2 = r\,\mathrm{d}r\mathrm{d}\theta\,. \tag{4.69}$$

This solution presents a naked singularity and, therefore, is prohibited by the cosmic censorship hypothesis. Unfortunately, all those solutions found in [57] have fields $\hat{\phi}^a$ and $\hat{\varphi}$ equal to zero, and thus all one-point functions (4.52) are zero. A similar conclusion was made in [57], rephrased in terms of charges. When deriving the one-point functions (4.52), we assumed that the spin-connection ω^{ab} is independent of the vielbein, and the result is meaningful only if we can find solutions with nonzero scalar fields and torsion. We leave this for future work, and in the future, we call the spin-current in CTG a putative spin-current.

4.2.2 Generalization to $D = 2n$ spacetime dimensions, for $n > 2$

Working along the same lines as before, we can write the CTG action in $D = 2n$ bulk dimensions as

$$\int_{\mathcal{M}_{2n}} \varepsilon_{CA_1B_1\ldots A_nB_n}\hat{\Phi}^C\hat{F}^{A_1B_1}\ldots\hat{F}^{A_nB_n} \equiv \int_{\mathcal{M}_{2n}} \varepsilon\hat{\Phi}\hat{F}^n\,. \tag{4.70}$$

In the last step, we suppressed the indices, and we shall continue to do this for the rest of this chapter. Following the results of [30], where the results of $5D$ CS gravity were lifted to a higher number of bulk dimensions, we can infer that the same form of FG expansion can be used as before. The on-shell variation of this action, using the FG gauge, takes the form

$$\begin{aligned}\delta S_{CTG}^{(2n)}|_{\mathrm{on-shell}} = \kappa\int_{\mathcal{M}_{2n}} \varepsilon\Big[\delta e\Big(4n\psi(R+4ek)^{n-1} + 8n(n-1)\varepsilon\phi\mathrm{D}k(R+4ek)^{n-2}\Big)\\ + \delta k\Big(- 4n\varphi(R+4ek)^{n-1} - 8n(n-1)\phi T(R+4ek)^{n-2}\Big)\\ + \delta\omega\Big(- 4n(n-1)\varphi\mathrm{D}k(R+4ek)^{n-2} + 4n(n-1)\psi T(R+4ek)^{n-2}\\ - 4n(n-1)\phi e^a k_a(R+4ek)^{n-2} + 8n(n-1)(n-2)\phi\mathrm{D}kT(R+4ek)^{n-3}\Big)\Big]\,.\end{aligned} \tag{4.71}$$

In order to remove the variation δk, we add finite GHY-like counterterm

$$S^{(2n)}_{GHY} = 4\kappa \int_{\mathcal{M}_{2n}} n \sum_{j=0}^{n-1} \frac{1}{n-j} \varepsilon k \varphi \binom{n-1}{j} R^j (4ek)^{n-1-j} + 2n(n-1) \sum_{j=0}^{n-2} \frac{1}{n-1-j} k\phi T \binom{n-2}{j} R^j (4ek)^{n-2-j} . \tag{4.72}$$

Once this boundary term is included, the one-point functions are obtained as before. The boundary stress-energy tensor is given by

$$\begin{aligned}\tau_a = \langle \mathcal{T}_a \rangle_{QFT} = \kappa\varepsilon \Big(& 4n\psi (R+4ek)^{n-1} + 8n(n-1)\phi \mathrm{D}k (R+4ek)^{n-2} \\ & + \sum_{j=0}^{n-1} \frac{4n}{n-j} k\varphi \binom{n-1}{j} (n-1-j) R^j e^{n-2-j} (4k)^{n-1-j} \\ & + \sum_{j=0}^{n-2} \frac{8n(n-1)}{n-1-j} \binom{n-2}{j} \mathrm{D}(k\phi R^j (4ek)^{n-2-j}) \\ & - \sum_{j=0}^{n-2} \phi T \frac{n-2-j}{n-1-j} \binom{n-2}{j} R^j (4k)^{n-2-j} e^{n-3-j} ,\end{aligned} \tag{4.73}$$

while a putative spin-current is

$$\begin{aligned}\sigma_{ab} = \langle \mathcal{S}_{ab} \rangle_{QFT} = \kappa \Big[\varepsilon \Big(& -4n(n-1)\varphi \mathrm{D}k (R+4ek)^{n-2} + 4n(n-1)\psi T (R+4ek)^{n-2} \\ & - 4n(n-1)\phi e^a k_a (R+4ek)^{n-2} + 8n(n-1)(n-2)\phi \mathrm{D}k T (R+4ek)^{n-3} \times \\ & \times 4n \sum_{j=0}^{n-1} \frac{1}{n-j} \binom{n-1}{j} j \mathrm{D}(k\varphi R^{j-1} (4ek)^{n-1-j}) \\ & + 8n(n-1) \sum_{j=0}^{n-2} \frac{1}{n-1-j} \binom{n-2}{j} j \mathrm{D}(k\phi T (4ek)^{n-2-j}) \Big) \\ & - 8n(n-1)\varepsilon \sum_{j=0}^{n-2} \frac{1}{n-1-j} \binom{n-2}{j} k\phi e R^j (4ek)^{n-2-j} \Big] .\end{aligned} \tag{4.74}$$

One-point function of an operator dual to scalar $\hat{\varphi}$ is

$$o_\varphi = \langle \mathcal{O}_\varphi \rangle_{QFT} = \kappa\varepsilon \Big(4n \sum_{j=0}^{n-1} \frac{1}{n-j} \binom{n-1}{j} k R^i (4ek)^{n-1-j} \Big) , \tag{4.75}$$

while the one-point function of operators dual to $\hat{\phi}^a$ yields

$$o_a = \langle \mathcal{O}_a \rangle_{QFT} = \kappa\varepsilon \Big(-8n(n-1) \sum_{j=0}^{n-2} \frac{1}{n-1-j} \binom{n-2}{j} k T R^j 4ek^{n-2-j} \Big) . \tag{4.76}$$

4.3 An aside: geodesics in gauge theories of gravity and boundary two-point function

So far in this section, we have been solely focused on a computation of one-point functions dual to bulk gravity fields (scalar, dilaton-like fields were treated as part of gravity). Of course, we should be able to do more, and for this, we should consider coupling this gauge theory of gravity to matter fields. Using the extrapolate dictionary (3.23), boundary correlation functions are obtained from the bulk one by a proper rescaling and taking the boundary limit. Propagators of matter fields are, for large values of mass, well-approximated by the exponent of a geodesic distance between two points, and therefore, there is a relation between bulk geodesics and boundary two-point functions. It is customary to define a two-point function using the path integral of the form

$$\int \mathcal{DP} e^{im\mathcal{L}} , \tag{4.77}$$

where $\mathcal{L}$ is the geodesic distance between two points, and we assume that this path integral can be evaluated in the saddle-point approximation to yield the boundary two-point function. It is then important to see if there is a natural candidate for the two-point function in the gauge-theoretic formulation of gravity. Here, we wish to argue that there is: the gravitational Wilson line (recal definition (1.20)).

Following [117], where the gravitational Wilson lines were introduced, with a difference of working in Euclidean signature, we write the Euclidean path integral as

$$\int \mathcal{DP}\mathcal{D}K\mathcal{D}h \, e^{-\int \mathrm{d}\tau \left(-\mathrm{Tr}(KA^h_\tau) + \lambda_1 \left(\frac{1}{2} J^{\hat{A}\hat{B}} J_{\hat{A}\hat{B}} - c_2 \right) + \lambda_2 \left(\frac{1}{16} \varepsilon_{\hat{A}\hat{B}\hat{C}\hat{D}} \varepsilon^{\hat{E}\hat{F}\hat{G}\hat{H}} J^{\hat{A}\hat{B}} J^{\hat{C}\hat{D}} J_{\hat{E}\hat{F}} J_{\hat{G}\hat{F}} - c_4 \right) \right)} , \tag{4.78}$$

where all the details about the notation can be found in the quoted paper. Our goal is to demon-

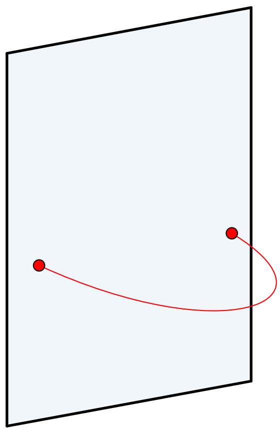

Figure 4.2: A graphical representation of a bulk gravitational Wilson line, giving a boundary two-point function.

strate that (4.78) can, in certain cases at least, reproduce the boundary two-point function for primary scalar operators (3.26). The proposed form of the path integral is a combination of the standard approach from (4.77) and the approach from [62], where the authors analysed the gravitational Wilson lines in AdS_3 gravity, using the CS formulation, in order to extract the boundary entanglement entropy. CS gravity in three dimensions is topological, and the Wilson line does not depend on local properties of the exact path taken between the two points. Therefore, the path integral measure $\mathcal{DP}$ was not necessary in [62]. Here, on the other hand, as shown in [117], different paths contribute differently, as the action is stationary only on a certain path, yielding the geodesics. An additional difference between [62] and our work in this section is that they were interested in computing the quantity that is expected to be infinite (cut-off dependent). We, therefore, should renormalize the infinite part of the two-point function, as standard in the extrapolate dictionary (3.23). In order to use the saddle-point approximation

in (4.78), we assume that the mass of a particle is large. Furthermore, we will consider only the spinless case, though we suspect that, at least for large nonzero values of spin, a similar computation can be done.
Varying the action from the path integral (4.78) with respect to h results in the equation

$$\frac{\mathrm{d}z^\mu}{\mathrm{d}\tau}p^\nu = \frac{\mathrm{d}z^\nu}{\mathrm{d}\tau}p^\mu\,, \tag{4.79}$$

which is obviously solved by $p^\mu = m\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$, with τ being the affine parameter. Varying with respect to the path $\mathcal{P}$ results in a geodesic motion of a probe particle. We shall illustrate the computation for the pure AdS spacetime in coordinates (1.32). Focusing on $t = $ const slices, it is well-known that the geodesics between two boundary points are semi-circles (see section 3.3). By focusing on the two points separated by a distance L at the boundary, and relying on the results from section 3.3, we obtain

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \lim_{\varepsilon\to 0}\varepsilon^{2m}e^{-S_{\text{on-shell}}} \sim \frac{1}{L^{2m}}\,. \tag{4.80}$$

This result matches the one obtained in (3.26), as for large mass we have

$$\Delta = \frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + m^2} \approx m\,. \tag{4.81}$$

4.4 Two dimensional BF model and JT gravity

EH gravity in two dimensions is trivial, as $\int \mathrm{d}^2x\sqrt{g}R$ computes the Euler characteristic of the manifold. Therefore, in order to obtain some nontrivial effects, we have to consider more general models. They usually demand the introduction of a new scalar field, called the dilaton, which very often arises from a dimensional reduction procedure. In case of a negative cosmological constant, a very simple model is given by JT gravity, with action

$$\frac{1}{16\pi G_N}\int_{\mathcal{M}_2}\mathrm{d}^2x\,\sqrt{-g}\,\varphi\big(R-\Lambda\big)\,. \tag{4.82}$$

Varying with respect to the dilaton, we obtain $R = \Lambda$. Note that φ serves as a Lagrange multiplier and, therefore, relation $R = \Lambda$ holds in the quantum theory. This is because the path integral over the dilaton field yields

$$\int\mathcal{D}\varphi\mathcal{D}g_{\mu\nu}e^{i\frac{1}{16\pi G}\int_{\mathcal{M}_2}\mathrm{d}^2x\sqrt{-g}\varphi\left(R-\Lambda\right)} \sim \int\mathcal{D}g_{\mu\nu}\delta(R-\Lambda), \tag{4.83}$$

where $\delta(R-\Lambda)$ is delta functional. For this reason, JT gravity is very popular, and many results in the quantum regime are known. Varying with respect to the metric field, we obtain another equation of motion

$$\nabla_\mu\nabla_\nu\varphi - g_{\mu\nu}\nabla^2\varphi + g_{\mu\nu}\varphi = 0\,. \tag{4.84}$$

If we further add some matter fields in the bulk, the right-hand side of (4.84) has to be supplemented with the matter stress-energy tensor. Action (4.82) has to be supplemented with the boundary GHY term to eliminate boundary variations of metric derivatives in the bulk direction. In the situation at hand, this term is given by

$$\frac{1}{8\pi G_N}\int_{\partial\mathcal{M}_2}\mathrm{d}y\sqrt{|h|}\phi K\,. \tag{4.85}$$

First-order formulation of this theory can be straightforwardly written using the action

$$S_1 = \kappa \int \varphi \varepsilon_{ab} \left(R^{ab} + e^a e^b\right) \,, \tag{4.86}$$

where $\kappa \equiv \frac{1}{16\pi G_N}$. In order to set torsion to zero, we introduce additional Lagrange multipliers ϕ^a such that the total action is given by

$$\kappa \int_{\mathcal{N}} \varepsilon_{ab} \left[\hat{\varphi}\left(\hat{R}^{AB} + \hat{e}^A \hat{e}^B\right) + 2\hat{\phi}^A \hat{T}^B\right] , \tag{4.87}$$

where the numerical factor 2 and the notation suitable for the holographic considerations are introduced for convenience. Equations of motion following from (4.87) are

$$\hat{R}^{AB} + \hat{e}^A \hat{e}^b = 0\,, \qquad \hat{T}^A = 0\,, \qquad \hat{\varphi}\hat{e}^A - \mathrm{D}\hat{\phi}^A = 0\,, \qquad \mathrm{d}\hat{\varphi} + \hat{\phi}^A \hat{e}_A = 0\,. \tag{4.88}$$

Action (4.87) is similar to the CTG action in four dimensions that we considered. Indeed, we can rewrite this action as

$$\kappa \int_{\mathcal{N}} \varepsilon_{\hat{A}\hat{B}\hat{C}} \hat{\phi}^{\hat{A}} \hat{F}^{\hat{B}\hat{C}}, \tag{4.89}$$

where $\hat{\phi}^{\hat{A}} = (\hat{\phi}^A, \hat{\varphi})$, $\hat{F}^{BC} = \hat{R}^{BC} + \hat{e}^B \hat{e}^C$ and $\hat{F}^{A3} = \hat{T}^A$. In this format, it is evident that our model is indeed $1+1$ D CTG, invariant under $SO(2,1)$ algebra. Equivalently, we can rewrite action (4.89) as follows. We introduce $SO(2,1) \approx SL(2,\mathbb{R})$ gauge field $A = \omega^{01} J_{01} + e^A P_A$, where J_{01} and P_A are $\mathfrak{so}(2,1)$ generators, given in representation as

$$P_0 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\frac{\sigma_3}{2}\,, \quad P_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\sigma_1}{2}\,, \quad J_{01} = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -\frac{\sigma_2}{2}\,. \tag{4.90}$$

For scalar multiplet Φ of the form $\Phi = \varepsilon^{AB} \hat{\phi}_B P_A + \hat{\varphi} J_{01}$, we have

$$\mathrm{Tr}\left(\Phi F\right) = \varphi \left(R^{ab} + e^a e^b\right) + 2\phi^a T^b \,, \tag{4.91}$$

so that the action (4.89) takes the form

$$\kappa \int \mathrm{Tr}(\Phi F)\,. \tag{4.92}$$

For the past decade, there has been increased interest in JT gravity due to its connection to the near-extremal limit of higher-dimensional black holes, its relation to the SYK model, and matrix models. Reviews on JT gravity include [118, 119].

It is important to understand which aspects of JT gravity, especially involving boundaries, can be fruitfully explained using this first-order formulation. Note the comment made in [119] where it was stated that for the full understanding of the boundary issues, one has to resort to a combination of first-order and second-order reasoning. However, it was emphasised in that text that there are works that tried to explore the power of first-order formalism in this theory. We also made contributions in this direction, and in this section, we will demonstrate how our approach can be applied to JT gravity. In addition to computing the BH mass and entropy from holography, we will also explore the dynamics of E.o.W branes, even though in such a low number of dimensions we are not interested in using AdS/BCFT duality. One main motivation for studying such branes in this theory comes from the fact that they played an important role in initial considerations on BH information paradox resolution [120].

Taking the on-shell variation of the bulk action, we have

$$\kappa \int_{\partial\mathcal{M}_2} \varepsilon_{AB}\big(\hat{\varphi}\delta\hat{\omega}^{AB} + 2\hat{\phi}^A\delta\hat{e}^B\big). \tag{4.93}$$

We will now plug in the same expansion for the bulk fields in terms of the radial ρ coordinate as before. Our goal here is to try to obtain the mass and entropy of a two-dimensional BTZ black hole using previously developed techniques. Focusing on finite terms, we have

$$4\kappa \int_{\partial\mathcal{M}_2} (-\varphi\delta k + \psi\delta e). \tag{4.94}$$

Again, we face the same problem as before. The solution we pose will be the same one we have already done. By adding a finite boundary term, we will remove the variation of δk at the boundary of our spacetime. Note that a similar setup was discussed in [121]. This is our choice of boundary conditions, motivated as before, and it is important to stress that it is by no means necessary to impose those boundary conditions. However, as we will see, our final result for the one-point function will provide satisfactory results regarding the BH thermodynamics. Therefore, we add a finite boundary term

$$S_{bnd} = 4\kappa \int_{\partial\mathcal{M}_2} \varphi k\,. \tag{4.95}$$

Because of the alleged connection of similar terms with GHY terms [79], we shall call this term a GHY-like boundary term. Later, we will explore in detail the relation with (4.82). Once this term is added, we obtain

$$\delta W = 4\kappa \int_{\partial\mathcal{M}_2} (k\delta\varphi + \psi\delta e)\,. \tag{4.96}$$

One-point functions of boundary operators are then inferred to be

$$\langle\mathcal{T}\rangle = 4\kappa\psi\,, \qquad \langle\mathcal{O}_\varphi\rangle = 4\kappa k\,. \tag{4.97}$$

Boundary fields are not completely independent and have to satisfy identities following from the bulk equations of motion

$$\mathrm{d}\varphi = e\phi\,, \quad \mathrm{d}\psi = k\phi\,, \quad \mathrm{d}\phi = 2k\varphi + 2e\psi\,. \tag{4.98}$$

4.4.1 Black hole thermodynamics

In this subsection, we shall use our results to analyze the thermodynamics of a two-dimensional black hole. Dimensional reduction of the spinless BTZ black hole gives

$$\mathrm{d}s^2 = -(r^2-\mu)\mathrm{d}t^2 + \frac{1}{r^2-\mu}\mathrm{d}r^2, \quad \hat{\varphi} = r. \tag{4.99}$$

We shall first spend a few lines describing this black hole. First, one may object that this is not a genuine black hole, but rather a portion of AdS_2. In a sense, we already know that any solution of JT gravity must satisfy the AdS condition. In three dimensions, the BTZ black hole indeed contains a singularity in the sense that geodesics cannot be extended to an infinite value of an affine parameter once they encounter $r = 0$. Two-dimensional black hole spacetime, being a portion of AdS_2, does not have this property. To illustrate this explicitly, we make the following change of variables [118]

$$r = \frac{-\mu T^2 + \mu Z^2 + 4}{4Z}\,, \qquad t = \frac{1}{2\sqrt{\mu}}\ln\left(\frac{\mu T^2 + 4\sqrt{\mu}T - \mu Z^2 + 4}{\mu T^2 - 4\sqrt{\mu}T - \mu Z^2 + 4}\right)\,. \tag{4.100}$$

One can then check that line element (4.99) becomes

$$\mathrm{d}s^2 = \frac{-\mathrm{d}T^2 + \mathrm{d}Z^2}{Z^2}\,, \tag{4.101}$$

which is a metric of AdS_2 in Poincaré coordinates (1.32). Nevertheless, we will call a spacetime described using coordinates (4.99) a $1+1$ BTZ black hole [122]. Another set of coordinates that covers the same region is (t, σ), with the line element given by

$$\mathrm{d}s^2 = \frac{4\pi^2}{\beta^2}\left(\frac{-\mathrm{d}t^2 + \mathrm{d}\sigma^2}{\sinh\frac{2\pi}{\beta}\sigma}\right)\,. \tag{4.102}$$

To obtain the BH mass, we shall first calculate the one-point function for the boundary energy-momentum tensor, which in this case (boundary is one-dimensional) gives us only the energy. We have

$$\frac{\delta W}{\delta e_t^t} = \frac{\delta W}{\delta g^{tt}}\frac{\partial(e_t^t e^{tt})}{\partial e_t^t} = -2\frac{\delta W}{\delta g^{tt}} = \langle \mathcal{T}\rangle = E\,. \tag{4.103}$$

Further, we go to the FG gauge, where the line element takes the form

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{4\rho^2} - \frac{1}{\rho}\Big(1 - \frac{\mu}{2}\rho + \frac{\mu^2}{16}\rho^2\Big)\mathrm{d}t^2\,, \tag{4.104}$$

and the corresponding boundary fields are

$$e = \mathrm{d}t\,, \quad k = -\frac{\mu}{4}\mathrm{d}t\,, \quad \varphi = 1\,, \quad \psi = \frac{\mu}{4}\,, \quad \phi = 0\,. \tag{4.105}$$

It is trivial to check that boundary fields satisfy constraints (4.98). Energy is then expressed as

$$E = 4\kappa\psi = \frac{\mu}{16\pi G}. \tag{4.106}$$

In the end, a simple usage of the first law of thermodynamics, $T\mathrm{d}\mathcal{S} = \mathrm{d}E$, and the condition that the entropy vanishes for zero BH mass, gives

$$\mathcal{S}_{JT} = \frac{\sqrt{\mu}}{4G}\,. \tag{4.107}$$

This entropy coincides with the usual form of entropy in JT gravity [118]. Note that this result can be written as

$$\mathcal{S}_{JT} = \frac{\varphi(r_h)}{4G}\,. \tag{4.108}$$

This is the expected form of Bekenstein-Hawking entropy in 2D dilaton gravity, generalizing (1.12). In two-dimensional models, the horizon consists of only two points, and the surface of an n-sphere ($\mathrm{Vol}(S^n) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$) approaches two as $n \to 1$. Therefore, for a single point, the natural generalization of area from (1.12) is a constant 1. However, in dilaton models, one way to interpret the dilaton is as a spacetime varying Newton's constant (as it multiplies the standard EH term, similar to $\frac{1}{G_N}$). Combining these two facts, we arrive at (4.108). Another point of view on (4.108) is that $\varphi(r)$ comes, via dimensional reduction, from the $e^2{}_2$ component of vielbein. In the original $3D$ BTZ black hole spacetime (2.65), this component was chosen to be equal to r, while the area of the horizon was given as $2\pi r_h = 2\pi e^2{}_2 = 2\pi r_h$, which again motivates (4.108). Finally, let us note that the equations of motion are linear in φ, so that another possible solution to the equations of motion can have the same metric, but $\varphi = ar$, with a an arbitrary constant. In this case, it is easy to see that the BH entropy modifies as $\mathcal{S}_{JT} = \frac{a\sqrt{\mu}}{4G}$, which again coincides with (4.108).

4.5 Boundary terms in two dimensions

Having established that our formalism for CTG can fruitfully derive the black hole entropy in JT gravity, we elaborate further on boundary terms in this two-dimensional model. Here we rely on the approach discussed in the subsection 2.3.2 and return to the standard notation used in this thesis. If we vary the action with respect to the spin-connection, a boundary term arises

$$\kappa \int_{\mathcal{N}} \left(2\varepsilon_{ab}\phi_c e^b - \varepsilon_{ac}\mathrm{d}\varphi\right) \delta\omega^{ac} + \kappa \int_{\partial\mathcal{N}} \varepsilon_{ab}\varphi\delta\omega^{ab} \,. \tag{4.109}$$

Variation with respect to the vielbein produces a boundary term proportional to δe^a, while variation with respect to the scalar field does not produce any boundary terms. Imposing Dirichlet boundary conditions for the vielbein, we see that the action is not stationary. While this may seem completely acceptable from the first-order perspective (and especially if we take the BF formulation into account), it is drastically different from the second-order perspective. This is similar to (2.54), so we take a similar solution. Moving the variation at the boundary from the spin-connection to the dilaton field, we add the following boundary term.

$$-\kappa \int_{\partial\mathcal{N}} \varepsilon_{ab}\varphi\omega^{ab} \,. \tag{4.110}$$

We should check if this term has all the properties we discussed in the subsection 2.3.2. First, let us rewrite this term using the FG expansion from the previous section.

$$\begin{aligned} -\int_{\partial\mathcal{N}} \varepsilon_{ab}\varphi\omega^{ab} &= -2\int_{\partial\mathcal{N}} \varphi\omega^{01} \\ &= -2\int_{\partial\mathcal{N}} \frac{1}{\rho}(\tilde{\varphi} + \rho\tilde{\psi})(\tilde{e} - \rho\tilde{k}) \,. \end{aligned}$$

We are interested in the ρ-finite part of this expansion

$$-2\int_{\partial\mathcal{N}} (-\tilde{\varphi}\tilde{k} + \tilde{\psi}\tilde{e}) = 4\int_{\partial\mathcal{N}} \tilde{\varphi}\tilde{k} \,. \tag{4.111}$$

To obtain the last equality, we used the bulk's equations of motion in the FG gauge (4.98) and removed the $\int \mathrm{d}\phi$ term, since it is a boundary term, and in the standard holographic setup, the boundary of the boundary is empty. Importantly, the result (4.111) matches (4.95), supporting the validity of the boundary term (4.110).

Again, we face the issue that the term (4.110) is not manifestly covariant, as it explicitly involves the spin-connection. We shall covariantize this expression as before, introducing a normalized vector field n^a normal to the boundary, to get

$$2\kappa \int_{\mathcal{Q}} \varepsilon_{ab}\varphi n^a \mathrm{D}n^b. \tag{4.112}$$

It is instructive to compare this first-order boundary term to the standard JT GHY boundary term (4.85). We have

$$\begin{aligned}
2\kappa \int_{\mathcal{Q}} \varepsilon_{ab}\varphi n^a \mathrm{D}n^b &= 2\kappa \int_{\mathcal{Q}} \varepsilon_{ab}\varphi e^a_\rho n^\rho \mathrm{D}_\mu (e^b_\nu n^\nu)\mathrm{d}x^\mu \\
&= 2\kappa \int_{\mathcal{Q}} \varepsilon_{ab}\varphi e^a_\rho n^\rho e^b_\nu \nabla_\mu n^\nu \mathrm{d}x^\mu \\
&= 2\kappa \int_{\mathcal{Q}} e\varepsilon_{\rho\nu}\varphi n^\rho \nabla_\mu n^\nu \mathrm{d}x^\mu \\
&= 2\kappa \int_{\mathcal{Q}} e\varepsilon_{\rho\nu}\varphi n^\rho \nabla_\mu n^\nu \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\mathrm{d}t\,.
\end{aligned} \tag{4.113}$$

Here, t parametrizes the one-dimensional boundary. The bulk metric can be decomposed as

$$g_{\mu\nu} = h_{\mu\nu} + n_\mu n_\nu\,, \tag{4.114}$$

where $h_{\mu\nu}$ is the projection tensor (sometimes referred to as the first fundamental form, or the induced metric) [54]. Furthermore, we can take $h_{\mu\nu} = -t_\mu t_\nu$, with t^μ being the normalized vector tangent to the boundary [123]. From (3.91), applied in the case at hand, we obtain $n^\mu \perp \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \equiv t^\mu$. Note that if t^μ is not normalized, the final expression should include an additional factor $\sqrt{h}$. We can define the Levi-Civita tensor $\overline{\varepsilon}_{\mu\nu} = e \cdot \varepsilon_{\mu\nu}$, which satisfies $\overline{\varepsilon}_{\mu\nu} n^\mu = -t_\nu$. To prove this, note that if we multiply both sides by n^ν, we get $0 = 0$. Therefore, $\overline{\varepsilon}_{\mu\nu} n^\mu$ has to be proportional to t_ν. Note that we have

$$\overline{\varepsilon}_{\mu\nu} n^\mu \overline{\varepsilon}_{\rho\sigma} n^\rho g^{\nu\sigma} = -\overline{\varepsilon}_{\mu\nu}\overline{\varepsilon}^\nu{}_\rho n^\mu n^\rho = g_{\mu\rho} n^\mu n^\rho = 1\,.$$

This fixes the proportionality constant to be ± 1, with two choices corresponding to two opposite orientations of the boundary. Finally, this implies

$$2\kappa \int_{\mathcal{Q}} \varphi t_\nu \nabla_\mu n^\nu \mathrm{d}x^\mu = 2\kappa \int_{\mathcal{Q}} \varphi t_\nu t^\mu \nabla_\mu n^\nu \mathrm{d}t = 2\kappa \int_{\mathcal{Q}} \varphi K\,, \tag{4.115}$$

so that we finally reached (4.112). This further supports the boundary term (4.110)

At the end of this section, we shall use the methodology of [31] to obtain the same boundary term as before. It follows from the analysis in [31] that only the $\int \varepsilon_{ab}\varphi R^{ab}$ term should be accompanied by the GHY boundary term. Using the same notation as in [31], we get

$$\star\varphi_{\mathbf{n}a}\delta\varrho^{\mathbf{n}a} \sim \kappa\varphi\varepsilon_{\mathbf{n}a}\delta\varrho^{\mathbf{n}a}\,, \tag{4.116}$$

resulting in

$$\star\varphi_{\mathbf{n}a} = \kappa\varphi\varepsilon_{\mathbf{n}a}\,. \tag{4.117}$$

This immediately gives the GHY boundary term as

$$2\kappa \int_{\partial\mathcal{Q}} \varphi K^a \varepsilon_{\mathbf{n}a} = 2\kappa \int_{\partial\mathcal{Q}} \varphi n^\mu D n^\nu \varepsilon_{\mu\nu}\,, \tag{4.118}$$

where the equality follows from the identity

$$e^a_\nu D n^\nu n^\mu \varepsilon_{\mu\rho} e^\rho_a = \varphi n^\mu D n^\nu \varepsilon_{\mu\nu}\,. \tag{4.119}$$

4.6 E.o.W. brane profile

After discussing the boundary terms, we are ready to compute the E.o.W. brane profile in this two-dimensional model of gravity. The total action of JT gravity plus E.o.W. brane with tension T is given by

$$\kappa \int_{\mathcal{N}} \varepsilon_{ab} \left[\varphi \left(R^{ab} + e^a e^b\right) - 2\mathrm{D}\phi^a e^b\right] + 2\kappa \int_{\mathcal{Q}} \varepsilon_{ab}\phi^a(\delta^b_c - n^b n_c)e^c + 2\kappa \int_{\mathcal{Q}} \varepsilon_{ab}\varphi n^a \mathrm{D} n^b \\ + 2\kappa T \int_{\mathcal{Q}} \varepsilon_{ab} n^a e^b \,. \tag{4.120}$$

Here, compared to (4.87), we used partial integration to get the second integral. We follow the section 3.4 and make the variation of the total action with respect to all the fields. Next, we impose Neumann boundary conditions on the E.o.W $\mathcal{Q}$. Let us start with the variation $\delta\omega$:

$$\kappa \int_{\mathcal{Q}} \varepsilon_{ab}\varphi\delta\omega^{ab} + 2\kappa \int_{\mathcal{Q}} \varepsilon_{ab}\varphi n^a \delta\omega^b{}_c n^c = 0 \,. \tag{4.121}$$

One can check that this variation is equal to zero due to the condition $n^a n_a = 1$. Alternatively, one may treat the field n^a as unnormalized, with this equation enforcing the normalization condition (though, in the latter case, we should appropriately normalize the boundary term first, as in [124]). Next, variation $\delta\phi^a$ trivially gives $n^a e_a|_{\mathcal{Q}} = 0$, which implies that n^a is, indeed, orthogonal to $\mathcal{Q}$. Again, we stress that this orthogonality is not postulated a priori, but rather follows from the boundary conditions, as in [106]. Next, we make the variation δe, providing us with the equation

$$\varepsilon_{ab}(T n^a + \phi^a) - \varepsilon_{ac}\phi^a n^c n_b = 0 \,, \tag{4.122}$$

that has to be satisfied. As a consistency check, if we multiply both sides of (4.122) by n_b, we get $0 = 0$. An important role is played by the variation in φ. Using the result (4.115) from the last section, we get the equation

$$\nabla_\mu n^\mu = 0 \,, \tag{4.123}$$

implying the E.o.W. takes the form of a geodesics [123]. Therefore, the result coincides with the one obtained in the metric formulation, see for example [120], where the E.o.W. brane in JT gravity was used to give a toy model for resolving the black hole information paradox. A similar result was also obtained in [125]. Finally, the variation δn gives

$$2\kappa \int_{\mathcal{Q}} \varepsilon_{ab}\varphi\delta n^a \mathrm{D} n^b + 2\kappa \int_{\mathcal{Q}} \varepsilon_{ab}\varphi \mathrm{D} n^b \delta n^a - 2\kappa \int_{\mathcal{Q}} \varepsilon_{ab} e^c \phi_c n^a \delta n^b \\ + 2\kappa T \int_{\mathcal{Q}} \varepsilon_{ab}\delta n^a e^b - 2\kappa \int_{\mathcal{Q}} \varepsilon_{ab}\phi^a n^b \delta n_c e^c \,.$$

Because of the identity

$$\varepsilon_{ab}\phi^c e_c n^a + \varepsilon_{ac}\phi^a n^c e_b = \varepsilon_{ab}\phi^b e_c n^c = 0 \,, \tag{4.124}$$

which follows from $e_c n^c|_{\mathcal{Q}} = 0$, we can recast the last variation as

$$2\kappa \int_{\mathcal{Q}} \varepsilon_{ab}\varphi\delta n^a \mathrm{D} n^b + 2\kappa \int_{\mathcal{Q}} \varepsilon_{ab}\varphi \mathrm{D} n^b \delta n^a + 2\kappa T \int_{\mathcal{Q}} \varepsilon_{ab}\delta n^a e^b \,. \tag{4.125}$$

Following [106], we multiply the equation following from (4.125) with the projector $\delta^a_b - n^a n_b$, because we have $\delta(n_a n^a) = 0$. The resulting equation takes the form

$$2\varepsilon_{ab}\varphi \mathrm{D} n^b + T\left(\varepsilon_{ab}e^b - n^c n_a \varepsilon_{cb} e^b\right) = 0 \,. \tag{4.126}$$

It turns out that this equation is automatically satisfied. Let us illustrate this fact. We first prove that $\varepsilon_{ab}e^b - n^c n_a \varepsilon_{cb} e^b = 0$. Assuming $a = 0$, we have

$$\begin{aligned} &\varepsilon_{0b}e^b - n^c n_0 \varepsilon_{cb} e^b \\ &= e^1 - n^0 n_0 \varepsilon_{01} e^1 - n^1 n_0 \varepsilon_{10} e^0 \\ &= e^1 - (1 - n^1 n_1) e^1 + n^1 n_0 e^0 \\ &= e^1 - (1 - n^1 n_1) e^1 - n^1 n_1 e^1 = 0\,. \end{aligned} \tag{4.127}$$

Here, we used the normalization identity $n_a n^a = 1$ and orthogonality condition $n^a e_a|_{\mathcal{Q}} = 0$. Similar computation is valid for $a = 1$. Indeed, we have

$$\varepsilon_{ab}e^b - n^c n_a \varepsilon_{cb} e^b = 0\,. \tag{4.128}$$

Furthermore, we wish to prove that $\mathrm{D}n^a = 0$. This follows from the goedesics condition $\varepsilon_{ab} n^a \mathrm{D} n^b = 0$. By writing down indices explicitly, we have

$$\begin{aligned} \varepsilon_{ab} n^a \mathrm{D} n^b &= n^0 \mathrm{d} n^1 - n^1 \mathrm{d} n^0 - \omega_{10}(n^0 n_0 + n^1 n_1) \\ &= n^0 \mathrm{d} n^1 - n^1 \mathrm{d} n^0 - \omega_{10} = 0. \end{aligned} \tag{4.129}$$

After multiplying this equation by n_0, we have

$$\begin{aligned} &n^0 n_0 \mathrm{d} n^1 - n^1 n_0 \mathrm{d} n^0 - \omega_{10} n_0 \\ &= (1 - n^1 n_1) \mathrm{d} n^1 - n^1 n_0 \mathrm{d} n^0 + \omega^1{}_0 n^0 \\ &= \mathrm{d} n^1 - \frac{1}{2} n^1 \mathrm{d}(n^a n_a) + \omega^1{}_0 n^0 \\ &= \mathrm{d} n^1 + \omega^1{}_0 n^0 = \mathrm{D} n^1 = 0. \end{aligned} \tag{4.130}$$

On the other hand, by multiplying (4.129) instead by n^1, we obtain

$$-\,\mathrm{d} n^0 + \frac{1}{2} n^0 (n_a n^a) - \omega_{10} n^1 = -\mathrm{D} n^0 = 0\,. \tag{4.131}$$

Therefore, all the terms in (4.126) vanish, and the variation in n^a is automatically satisfied.

Chapter 5

Chern-Simons gravity with torsion

This chapter is partially based on author's work [67].

Classically, $3D$ gravity is described, modulo subtleties, as CS theory for an appropriate gauge group (Poincaré, dS, or AdS), see section 2.3. We also saw that the same formalism can be used to include torsion in the bulk, yielding the MB theory. On the other hand, $5D$ CS gravity is also closely related to torsion, because of the BH solutions (2.81) we discussed in Chapter 2. Both three-dimensional gravity with torsion and five-dimensional gravity with torsion have been studied holographically in the past [26, 25]. For this reason, we found it obligatory to extend the analysis to an even number of dimensions in the last chapter, leaving aside for the moment the odd-dimensional case. However, it is far from being true that we completely understand the role of torsion in odd-dimensional CS gravity. For example, can we extend the E.o.W. brane computation from section 3.4 to incorporate bulk torsion? Can we generalize the RT proposal to compute boundary entanglement entropy when the bulk has nonzero torsion? Does the parameter C from (2.81) enter the expressions for the BH energy and entropy, and does it affect the holographic entanglement entropy? Can we use dimenisonal reduction procedure to gain some insight into these questions? The following chapter precisely pursues those questions we just asked.

5.1 AdS/BCFT with torsional bulk

In Chapter 2, we defined the MB model (2.36) and showed it leads to the Riemann-Cartan bulk. The torsion in the model is influenced by two constants α_3 and α_4, and when both of them vanish, we are left with EH gravity with the cosmological constant, so that the bulk geometry is Riemannian. As anticipated before, it is much simpler to study the case $\alpha_3 = 0$, so we focus on this case in the following. Later, we will comment on why this choice is not just a simplification, but also the necessity for some of the calculations. We, therefore, focus on the action

$$\kappa \int_{\mathcal{N}} \varepsilon_{abc} \left(R^{ab} e^c + \frac{1}{3} e^a e^b e^c \right) + \alpha\kappa \int_{\mathcal{N}} T^a e_a \,, \tag{5.1}$$

where $\alpha \equiv \alpha_4$, and try to generalize the AdS/BCFT proposal to the Riemann-Cartan bulk spacetime. We put the lenght scale, present in the cosmological constant term, $\ell = 1$. Because of the translational CS term, even a simple AdS spacetime without torsion is not a solution to the equations of motion. The equations, following from the variation of (5.1) with respect to

e^a and ω^{ab} take the form of

$$R^{ab} + \left(1-\alpha^2\right) e^a e^b = 0, \tag{5.2}$$
$$T^c = -\frac{\alpha}{2}\varepsilon^{abc} e_a e_b. \tag{5.3}$$

Of course, the spacetime obtained in this way satisfies the AdS condition, and we can try to solve the equations by assuming that the metric takes the form of AdS spacetime in Poincaré coordinates (1.32). Actually, following [106], we will take the line element in the form of

$$ds^2 = e^{2\rho}(-\mathrm{d}t^2 + \mathrm{d}\phi^2) + l^2 \mathrm{d}\rho^2\,. \tag{5.4}$$

We made the change of variables $z = e^{-\rho}$ and introduced the parameter l (not to be confused with $\ell = 1$). The vielbeins are then given by

$$e^0 = e^\rho \mathrm{d}t\,, \quad e^1 = l\mathrm{d}\rho\,, \quad e^2 = e^\rho \mathrm{d}\phi \tag{5.5}$$

However, the spin-connection is not directly derived from the metric, and we have to use (2.41) to get

$$\begin{aligned}
\omega^{01} &= e^\rho\left(\frac{1}{l}\mathrm{d}t - \frac{\alpha}{2}\mathrm{d}\phi\right)\,,\\
\omega^{02} &= \frac{l\alpha}{2}\mathrm{d}\rho\,,\\
\omega^{12} &= e^\rho\left(-\frac{\alpha}{2}\mathrm{d}t - \frac{1}{l}\mathrm{d}\phi\right)\,.
\end{aligned} \tag{5.6}$$

The Riemann-Cartan curvature computed from this spin-connection gives

$$R^{ab} = \left(\frac{\alpha^2}{4} - \frac{1}{l^2}\right) e^a e^b\,, \tag{5.7}$$

so that the direct comparison with (5.2) reveals the connecton between the constant l and parameter α as

$$\frac{1}{l^2} = 1 - \frac{3\alpha^2}{4}\,. \tag{5.8}$$

We assume that $1-\frac{3\alpha^2}{4} > 0$. Next, we should discuss the boundary terms in this model that we shall use to derive the E.o.W. brane profile. Following Figure 3.3, boundary of the bulk $\partial\mathcal{N}$ has two components $\partial\mathcal{N} = \mathcal{M}\cup\mathcal{Q}$. Our main focus will be on the E.o.W. brane profile $\mathcal{Q}$, so we will analyse this boundary in what follows. Regarding the boundary $\mathcal{M}$, we treat it as standard in the holographic duality, imposing the Dirichlet boundary conditions for holographic sources.

First, we make the on-shell variation of (5.1), resulting in the boundary term,

$$-\kappa\int_{\mathcal{Q}} \varepsilon_{abc} e^a \delta\omega^{bc} - \alpha\kappa\int_{\mathcal{Q}} e_a \delta e^a. \tag{5.9}$$

The first term contains the boundary variation $\delta\omega$, which we try to remove by adding an appropriate boundary term to (2.36). Luckily, there exists such a boundary term (which has the same structure as in the first-order EH gravity [126]) and is given by

$$\kappa\int_{\mathcal{Q}} \varepsilon_{abc} e^a \omega^{bc}\,. \tag{5.10}$$

Analogously to the (2.56), we should covariantize this term. We will now explicitly go over the computations necessary to perform this step, filling in all the blanks made when previously using heuristic arguments to perform the covariantization

$$\begin{aligned}
&-2\kappa\int_{\mathcal{Q}}\varepsilon_{abc}e^a n^b \mathrm{D}n^c \\
&=-2\kappa\int_{\mathcal{Q}}\epsilon_{abc}e^a n^b \mathrm{d}n^c-2\kappa\int_{\mathcal{Q}}\epsilon_{abc}e^a n^b\omega^{cd}n_d \\
&=-2\kappa\int_{\mathcal{Q}}\epsilon_{abc}e^a n^b \mathrm{d}n^c+\kappa\int_{\mathcal{Q}}e^a(\delta^b_a-n_a n^b)\varepsilon_{bcd}\omega^{cd}\,.
\end{aligned}\tag{5.11}$$

In the last step, we used $\omega_a=\frac{1}{2}\varepsilon_{abc}\omega^{bc}$, implying $\omega^{ab}=-\varepsilon^{abc}\omega_c$ [126] such that the following identity holds

$$\begin{aligned}
-2\varepsilon_{abc}e^a n^b\omega^{cd}n_d &= 2\varepsilon_{abc}e^a n^b\varepsilon^{cdm}\omega_m n_d \\
&= -2e^a n^b(\omega_b n_a-\omega_a n_b) \\
&= 2e^a(n^2\delta^b_a-n_a n^b)\omega_b \\
&= e^a(\delta^b_a-n_a n^b)\varepsilon_{bcd}\omega^{cd}\,.
\end{aligned}\tag{5.12}$$

In the computations, we used $n^2=n_a n^a=1$, and the combination $P^b_a=\delta^b_a-n_a n^b$ is the projector already encountered in previous chapters. Next, we should check if this term coincides with the boundary term introduced to satisfy the holographic boundary conditions in [26]. The FG gauge can be chosen in the same form as (3.56) and the boundary term from [26], in the case $\alpha_3=0$, is given by

$$4\kappa\int_{\partial\mathcal{M}}\varepsilon_{ij}\tilde{e}^i\tilde{k}^j\,.\tag{5.13}$$

We would like to compare this term on-shell to the (5.10). It is easy to see that, once expansion (3.56) is inserted into (5.10), the finite term vanishes. On the other hand, bulk equations of motion imply the identity

$$\mathrm{d}\tilde{\omega}^{ij}+2\tilde{e}^i\tilde{k}^j+2\tilde{k}^i\tilde{e}^j=0\,,\tag{5.14}$$

implying further that the boundary term (5.13) vanishes (the same logic is applied in the case of JT gravity (4.111)). Therefore, we made the connection between those two boundary terms, further motivating our choice (5.10). Finally, note that we would obtain the same boundary term using the methodology of [31], as the translational CS term, involving only torsion as the field strength, does not contribute to the GHY boundary terms. Combining all the terms into a single action, we are left with

$$\begin{aligned}
&\kappa\int_{\mathcal{N}}\varepsilon_{abc}\left(R^{ab}e^c+\frac{1}{3}e^a e^b e^c\right)+\alpha\kappa\int_{\mathcal{N}}T^a e_a-2\kappa\int_{\mathcal{Q}}\varepsilon_{abc}e^a n^b\mathrm{d}n^c \\
&+\kappa\int_{\mathcal{Q}}e^a(\delta^b_a-n_a n^b)\varepsilon_{bcd}\omega^{cd}+\kappa T\int_{\mathcal{Q}}\varepsilon_{abc}n^a e^b e^c\,.
\end{aligned}\tag{5.15}$$

Performing the variation $\delta\omega$ we get the equation

$$e_a n^a|_{\mathcal{Q}}=0\,,\tag{5.16}$$

enforcing the orthogonality condition for $n^a=n^\mu e^a_\mu$. Further variation δn^a results in an identity, if we assume that $n_a n^a=1$, which is the same case we encountered in section 3.4 and (4.6). Variation δe results in a nontrivial condition that has to be satisfied by the brane profile

$$P^b_a\varepsilon_{bcd}\omega^{cd}-2\varepsilon_{abc}n^b\mathrm{d}n^c+2T\varepsilon_{abc}n^b e^c+\alpha e_a=0\,.\tag{5.17}$$

We have to solve this equation to find the brane's profile. We make an ansatz for the profile as $\phi = g(\rho)$, implaing the condition $(\mathrm{d}\phi - g'(\rho)\mathrm{d}\rho)|_{\mathcal{Q}} = 0$. It is straightforward to find the components of the unit normal (with the Lorentzian index) as

$$\begin{aligned} n^0 &= 0\,, \\ n^1 &= -\frac{g'(\rho)}{l\sqrt{\frac{(g'(\rho))^2}{l^2} + e^{-2\rho}}}\,, \\ n^2 &= \frac{e^{-\rho}}{\sqrt{\frac{(g'(\rho))^2}{l^2} + e^{-2\rho}}}\,. \end{aligned} \tag{5.18}$$

We shall first be interested in $a = 2$ component of (5.17). We rely on the following identity

$$n^a\omega_a = -\frac{\alpha}{2}n^a e_a - \frac{1}{l}n^0 e^2 + \frac{1}{l}n^2 e^0 = -\frac{1}{l}n^2 e^0\,, \tag{5.19}$$

which follows from the fact that $n^0 = 0$ and the orthogonality condition $n^a e_a|_{\mathcal{Q}} = 0$, to obtain

$$g(\rho) = \pm\frac{l^2 T}{\sqrt{1 - l^2 T^2}}e^{-\rho} + \text{const}\,. \tag{5.20}$$

Next, we consider the $a = 1$ component of (5.17), which implies that the plus sign should be used in (5.20). With this in mind, we have $n^1 = lT$ and $n^2 = \sqrt{1 - l^2T^2}$. The final component of (5.17), $a = 0$, is satisfied automatically, so that the E.o.W. brane profile is given by

$$\phi = \frac{l^2 T}{\sqrt{1 - l^2 T^2}}e^{-\rho} + \text{const}\,. \tag{5.21}$$

To remind ourselves, we have $\frac{1}{l^2} = 1 - \frac{3\alpha^2}{4}$, and only through this parameter, the constant α, that is responsible for nonzero bulk torsion, enters the brane's profile. This result is a consequence of a relatively simple bulk model of gravity, as we shall discuss in the next section. However, as far as we are aware, this is the first incarnation of the Riemann-Cartan bulk in the AdS/BCFT correspondence and constitutes one of the major deliverables from this chapter.

5.2 RT formula for bulk with torsion

Motivated by our success to apply the AdS/BCFT setup to the MB model, we now turn to understanding the possible generalization of the RT formula (3.80) to the Riemann-Cartan bulk, described using action (5.1). Let us first comment on the boundary conformal symmetry in this model. It was shown in [127] that the boundary dual to the MB model (2.36) has two unequal central charges, given by

$$c_L = 24\pi\left(\frac{l}{16\pi G_N} + \alpha_3\left(\frac{Cl}{2} - 1\right)\right)\,, \qquad c_R = 24\pi\left(\frac{l}{16\pi G_N} + \alpha_3\left(\frac{Cl}{2} + 1\right)\right)\,, \tag{5.22}$$

where $l^{-2} = \frac{C^2}{4} - B$ (which coincides with (5.8) when $\alpha_3 = 0$). Adding a bulk CS term for the spin-connection, therefore, implies that $c_L \neq c_R$, which in turn means that the boundary CFT suffers from a gravitational anomaly [128]. As long as we are not willing to couple this theory to gravity (and in the context of AdS/CFT, boundary theory is naturally defined without dynamical gravity), this is not a problem, though it has certain implications. For us, it is important that no BCFT, with appropriate boundary conditions, can be formulated for a theory with a gravitational anomaly [129]. This is precisely why it was necessary to restrict ourselves in the previous section to the case of $\alpha_3 = 0$. This particular case was also of interest in initial studies of $3D$ AdS gravity with torsion in [130, 131], and a further choice of the parameter α_4, such that $B = 0$, provides us with the teleparallel formulation of the three-dimensional gravity. Let us, therefore, continue our study of the $\alpha_3 = 0$ case. From (5.22), conformal symmetry at the boundary has two equal central charges $c_L = c_R = \frac{3l}{2G_N}$, as in the Riemannian case (the difference being that l is not directly related to the constant multiplying the cosmological constant term in the action (5.1)). We should then be able to derive an analogue of the RT formula, which enables one to compute the entanglement entropy of a CFT interval using geometric bulk quantities. More precisely, we expect that the entanglement entropy in a dual field theory should be given by (3.79), or in other words

$$S_{EE} = \frac{3}{2G_N\left(1 - \frac{3\alpha^2}{4}\right)} \ln\left(\frac{L}{\varepsilon}\right) . \tag{5.23}$$

Unfortunately, as the bulk is described using Riemann-Cartan geometry, no direct generalisation has been discussed in the literature so far.

The constant l appearing in the central charges is ultimately related to the Riemannian part of the geometry. It appears in the metric (5.4), and it turns out that this constant corresponds to the AdS radius computed if we forget about the affine structure of the spacetime and focus solely on the metric structure. Additionally, we can see that from the bulk's point of view, the role of the translational CS term is precisely to change the Riemannian cosmological constant. Let us illustrate this. We start from the action (5.1) and equations (2.39) that give an algebraic relation between the torsion and the vielbein. We substitute the algebraic torsion back into the action. Further, we decompose the Riemann-Cartan curvature as

$$R^{ab} = \tilde{R}^{ab}(e) + \mathrm{D}K^{ab} + K^a{}_c K^{cb} , \tag{5.24}$$

with K^{ab} being the contorsion one-form. Here, we have $C = -\alpha$, $K^{ab} = -\frac{\alpha}{2}\varepsilon^{abc}e_c$. We also express the RC curvature from (5.1) in terms of the Riemann curvature and constant α. This yields the action

$$\kappa\int_{\mathcal{N}}\varepsilon_{abc}\left(\tilde{R}^{ab}e^c + \frac{\alpha^2}{4}e^a e^b e^c + \frac{1}{3}e^a e^b e^c - \frac{\alpha^2}{2}e^a e^b e^c\right) = \kappa\int_{\mathcal{N}}\varepsilon_{abc}\left(R^{ab}e^c + \frac{1}{3l^2}e^a e^b e^c\right) . \tag{5.25}$$

Therefore, we reached the conclusion that, as far as the classical description of the bulk theory is important, this model is equivalent to the Riemannian theory with a shifted cosmological constant. This further supports our claim (5.23).

However, the question of whether this claim can be derived directly in the case of the RC bulk (in the first-order formalism) remains unanswered. We wish to argue that this claim can, indeed, be derived using reasonable assumptions. To do so, we closely follow reasoning from [62], mentioned in the section (3.3). In this paper, the RT proposal was recast in three-dimensional EH gravity as the computation of the gravitational Wilson line anchoring boundary, recal section 4.3 for the introduction of the gravitational Wilson lines. For AdS_3 bulk, Wilson lines are labeled by the irreducible representations of $\mathrm{sl}(2,\mathbb{R})\oplus\mathrm{sl}(2,\mathbb{R})$ algebra. As we shall also need this later in the thesis, we now state some important facts about the unitary irreducible representations of $\mathrm{sl}(2,\mathbb{R})$ (see also Appendix A.2 for details). Due to the noncompactness of the Lie group, all unitary irreducible representations are infinitely dimensional [132]. They are classified by the value of a quadratic Casimir operator as belonging to

1. principal continuous series,
2. complementary continuous series,
3. discrete series.

Not all unitary irreducible representations are highest/lowest weight representations. If we choose the discrete series representations, fortunately, they are of this type. Furthermore, representations of $\mathrm{sl}(2,\mathbb{R}) \oplus \mathrm{sl}(2,\mathbb{R})$ are constructed using the tensor product of $\mathrm{sl}(2,\mathbb{R})$ representations.

As in (4.78), the trace in the Wilson loop is performed using the path integral over K and h (as equations of motion imply $F = 0$, the exact position of a Wilson line is not important, as long as it does not wrap a nontrivial cycle, in the presence of a black hole, for example, so that we do not require the integration over paths $\mathcal{P}$). In was shown in [62] that, relying on completely general considerations on the CS theory, the saddle point approximation for the path integral gives the following equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left((A-\bar{A})_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right) + [\bar{A}_\mu, A_\nu]\,\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0\,, \tag{5.26}$$

with s parametrizing the path. To be more precise, to derive this equation, it was assumed that the appropriate boundary condition at the CFT boundary is $h = 1$; we take the same assumption here. We assume that the same reasoning from [62] can be applied to (5.1), with the only distinction being the choice of the $\mathrm{sl}(2,\mathbb{R})$ gauge connection (2.42). In the case at hand, we have $q = \frac{\alpha}{2} + \frac{1}{l}$ and $\overline{q} = \frac{\alpha}{2} - \frac{1}{l}$. This further implies

$$A^a = \tilde{\omega}^a(e) - \frac{\alpha}{2}e^a + \left(\frac{\alpha}{2} + \frac{1}{l}\right)e^a\,, \qquad \overline{A}^a = \tilde{\omega}^a(e) - \frac{\alpha}{2}e^a + \left(\frac{\alpha}{2} + \frac{1}{l}\right)e^a\,, \tag{5.27}$$

so that when (5.26) is expressed in terms of $\tilde{\omega}^a(e)$, it gives the same equation as in the case of EH gravity. This equation provides us precisely with the geodesics, as if we completely neglect the affine structure of the spacetime. Furthermore, it is easy to check that the saddle-point approximation of the path integral from [62] then again yields the geodesic length, thus reproducing the result (5.23).

5.3 $2D$ reduction of $5D$ CS gravity with torsion

This section is based on the author's yet unpublished work.

Higher-dimensional theories of gravity are hard to study. This is true for EH gravity, but also for more general gravity theories. While CS gravity has certain nice properties, enabling us to find even the exact solutions with torsion, the full equations (2.33) and (2.34) are still very complicated. For this reason, it is fruitful to dimensionally reduce the theory to a lower number of dimensions. In this chapter, we will be interested in the dimensional reduction over $\mathbb{S}^3$, thereby reaching an effective two-dimensional theory. Most of the work will be analogous to the beginning of Chapter 4. The most important result from this chapter will be a formula demonstrating how torsion affects the entropy of the CS black hole (3.76).

5.4 Reduction procedure

Going through the literature, most attempts to perform the dimensional reduction assume the Riemannian geometry. For example, in [133], the dimensional reduction of five-dimensional Lovelock theory was considered, obtaining the effective two-dimensional Riemannian gravity. In that paper, the reduction was done over $\mathbb{R}^3$. Here, we make two differences. First, we do reduction over $\mathbb{S}^3$. This means that we shall be working with the metric of the form

$$g = \begin{pmatrix} g_{\mu\nu} & 0 & 0 & 0 \\ 0 & \left(\frac{e^{-\varphi}}{\lambda}\right)^2 & 0 & 0 \\ 0 & 0 & \left(\frac{e^{-\varphi}}{\lambda}\right)^2 \sin^2\psi & 0 \\ 0 & 0 & 0 & \left(\frac{e^{-\varphi}}{\lambda}\right)^2 \sin^2\psi \sin^2\theta \end{pmatrix}, \tag{5.28}$$

where $g_{\mu\nu}$ is a two-dimensional metric. The second difference is that we will work in the realm of Riemann-Cartan spacetime. In the spirit of first-order formalism, we have to make the reduction ansatz for both the vielbein and the spin-connection. Concerning the vielbein, (5.28) directly dictates their choice. They are given by

$$E^a = e^a_0 \mathrm{d}t + e^a_1 \mathrm{d}r \equiv e^a, \qquad E^i = \frac{e^{-\varphi}}{\lambda} e^i_S, \tag{5.29}$$

with the same convention as in (3.69). Compared to the dimensional reduction in Chapter 4, we made two differences. First, we used the form $e^{-\varphi}$ instead of φ, as the former one is more common in the literature (and in the five-dimensional picture provides that the radius of $\mathbb{S}^3$ is positive). Furthermore, we introduced a dimensionful parameter λ that was previously set to be equal to the AdS radius ℓ. The ansatz for the spin-connection is more involved. Generically, we obtained (5.28) as the most general spherical symmetric metric, so here we will rely on the results obtained in [81] for the most general, spherical symmetric and stationary spacetime with torsion. It is given by

$$\begin{aligned} &\Omega^{01} = A(r,t)\mathrm{d}r + B(r,t)\mathrm{d}t \equiv \omega^{01}, \quad \Omega^{0i} = \phi^0 e^i_S, \quad \Omega^{1i} = \phi^1 e^i_S, \\ &\Omega^{23} = -\cos\psi \mathrm{d}\vartheta + \Phi e^4_S, \quad \Omega^{24} = -\cos\psi\sin\vartheta \mathrm{d}\varphi - \Phi e^3_S, \quad \Omega^{34} = -\cos\vartheta \mathrm{d}\varphi + \Phi e^2_S. \end{aligned} \tag{5.30}$$

We introduced new fields ϕ^a, φ, and Φ, which all play the role of dilaton-like fields. Let us interpret all those fields. First, φ is the standard dilaton field, from metric (5.28). Fields ϕ^a constitute a Lorentz vector multiplet. We can prove this fact analogously to the derivation from the Chapter 4.

First, we make the gauge transformation of the connection (1.16) $A \to A - \mathrm{D}\epsilon$, for $\epsilon = \frac{1}{2}\epsilon^{AB} J_{AB} + \epsilon^A P_A$. In order to analyse the transformations of fields under two-dimensional Lorentz transformations, we take $\epsilon = \varepsilon J_{01}$. It is then easy to compute

$$\delta A = -\mathrm{d}\varepsilon J_{01} - \varepsilon \left(\Omega^{0i} J_{i1} + \Omega^{i1} J_{i0} - \frac{1}{\ell} E^0 P_1 + \frac{1}{\ell} E^1 P_0 \right), \tag{5.31}$$

implying that we have

$$\begin{aligned} A_\varepsilon = &\left(\omega^{01} - \mathrm{d}\varepsilon\right) J_{01} + \left(\Omega^{0i} + \varepsilon\Omega^{i1}\right) J_{0i} + \left(\Omega^{1i} + \varepsilon\Omega^{0i}\right) J_{1i} \\ &+ \frac{1}{2}\Omega^{ij} J_{ij} + \frac{1}{\ell}\left(e^0 - \varepsilon e^1\right) P_0 + \frac{1}{\ell}\left(e^1 + \varepsilon e^0\right) P_1. \end{aligned} \tag{5.32}$$

Comparing this formula with the general expansion in (1.16), we obtain the following transformation law for ϕ^a

$$\phi^0 e^i \to \phi^0 e^i - \varepsilon\phi^1 e^i\,, \qquad \phi^1 e^i \to \phi^1 e^1 + \varepsilon\phi^0 e^i\,, \tag{5.33}$$

or, more compactly written as

$$\phi^a \to \phi^a + \varepsilon\varepsilon^a{}_b\phi^b\,. \tag{5.34}$$

Equation (5.32) also gives the correct transformation property of the two-dimensional vielbein and spin-connection. Note that when computing the gauge transformations for (1.16), we are not generating the term that would normally be present in the variation of the form under local transformations in Poencare gauge theory. This is consistent with the comments made in [34]. Finally, a very important field Φ enters the spin-connection on the $\mathbb{S}^3$ part of the total manifold $\mathcal{M}_2 \times \mathbb{S}^3$. This field modifies the Riemannian sphere into a more general Riemann-Cartan manifold, while preserving spherical symmetry. To gain further intuition, we can compare our ansatz to the spherical solution of $5D$ CS gravity (2.73) with torsion. From the form of torsion (2.81) and the respective vielbeins, it follows that the contorsion one form K^a can be written as

$$K^{01} = 0\,, \quad K^{0i} = K^{1i} = 0\,, \quad K^{ij} = -C\varepsilon^{ijk}e^k_S\,. \tag{5.35}$$

It is evident that the axial torsion C corresponds to the field Φ (up to a global minus sign). Further analysis of the black hole (2.73) shows that the effective two-dimensional manifold is Riemannian, and that additional fields ϕ^a are obtained from the derivative of the dilaton field φ, while the dilaton field itself is equal to $-\ln(\lambda r)$. More concretely, if we introduce a two-dimensional vector field $\phi_\mu = e^a_\mu\phi_a$, we have

$$\phi_\mu = \frac{e^{-\phi}}{\lambda}\nabla_\mu\phi\,. \tag{5.36}$$

Generally, assuming that the five-dimensional geometry is Riemannian corresponds precisely to the choice (5.36) and $\Phi = 0$.

Using the reduction ansatz (5.30), we can compute the components of the curvature two-form. For example, we have

$$\begin{aligned} R^{02} &= \mathrm{d}\Omega^{02} + \Omega^0{}_1\Omega^{12} + \Omega^0{}_3\Omega^{32} + \Omega^0{}_4\Omega^{42} \\ &= \mathrm{d}\phi^0\mathrm{d}\psi + \omega^0{}_1\phi^1\mathrm{d}\psi - \phi^0\Phi\left(e^3e^4 - e^4e^3\right)\,, \end{aligned} \tag{5.37}$$

which generalizes to

$$R^{ai} = (\mathrm{D}\phi^a)e^i - \Phi\phi^a\varepsilon^{ijk}e^je^k\,. \tag{5.38}$$

Next, we compute

$$R^{23} = \mathrm{d}\Omega^{23} + \Omega^2{}_0\Omega^{03} + \Omega^2{}_1\Omega^{13} + \Omega^2{}_4\Omega^{43}\,, \tag{5.39}$$

which further generalizes to

$$R^{ij} = -(1 + \phi^a\phi_a)\hat{e}^i\hat{e}^j + \mathrm{d}\phi\varepsilon^{ijk}\hat{e}^k\,. \tag{5.40}$$

Finally, we have

$$R^{01} = \mathrm{d}\Omega^{01} + \Omega^0_i\Omega^{i1} = \mathrm{d}\omega^{01}\,. \tag{5.41}$$

We now have all the ingredients to start the dimensional reduction procedure, analogous to the one in Chapter 4. We shall not go through the details of the computations, as they are similar to those already written in this thesis, but will simply write down the final expression for the reduced action as

$$\begin{aligned} S_{\text{reduced}} = \frac{k\pi^2}{\ell^3\lambda^3}\int \varepsilon_{ab}\, e^{-3\varphi}\Big[\big(1+3\ell^2\lambda^2e^{2\varphi}(1-\phi_a\phi^a)\big)R^{ab} + \big(3\ell^{-2}+3\lambda^2e^{2\varphi}(1-\phi_a\phi^a)\big)e^ae^b \\ + \big(6\lambda e^{\varphi}+6\ell^2\lambda^3e^{3\varphi}(1-\phi_a\phi^a)\big)e^a\mathrm{D}\phi^b + 6\ell^2\lambda^2e^{2\varphi}\mathrm{D}\phi^a\mathrm{D}\phi^b\Big] \\ + \frac{k\pi^2}{\ell}\int \varepsilon_{ab}\Phi^2\left[-3\lambda^{-1}e^{-\varphi}\left(R^{ab}+\ell^{-2}e^ae^b\right)+6\phi^aT^b\right] - \frac{6k\pi^2}{\ell}\int_{\partial}\varepsilon_{ab}\Phi^2\phi^ae^b\,, \end{aligned} \tag{5.42}$$

This action is the first important result of this section. We shall call the theories described by this action a dimensionally reduced CS theory (DRCS).

5.4.1 JT term

Let us make an interesting observation about (5.42). CS gravity has a nice property that it is consistent to reduce itself to the torsionless sector. In this case, we could write down the action in the second-order formulation, using (5.36), and $\Phi = 0$. However, we already saw interesting solutions in five dimensions with nonzero Φ. Therefore, we focus on the term containing precisely this field. Interestingly, this term takes the form

$$\frac{k\pi^2}{\ell}\int \varepsilon_{ab}\Phi^2\left[-3\lambda^{-1}e^{-\varphi}\left(R^{ab}+\ell^{-2}e^ae^b\right)+6\phi^aT^b\right]\,. \tag{5.43}$$

Furthermore, if we assume that the two-dimensional manifold is torsionless, a more useful form of the previous term is given by

$$-\frac{3k\pi^2}{\lambda\ell}\int \mathrm{d}^2x\,\sqrt{-g}\Phi^2e^{-\varphi}\left(R+\frac{2}{\ell^2}\right). \tag{5.44}$$

Varying with respect to the field Φ yields two possible options

$$\Phi = 0 \qquad \text{or} \qquad R = -\frac{2}{\ell^2}. \tag{5.45}$$

This is precisely the action for JT gravity. Thus, the effect of axial torsion is to couple JT gravity with the Riemannian DRCS gravity. Using this interpretation, we can obtain the formula for the black hole entropy. To prepare the stage for this computation in the following subsection, note that using the Euclidean path integral formalism, it is evident that the entropy of a black hole with a nonzero Φ field is given by the expression for $\Phi = 0$ plus the standard JT gravity result. The latter one is given by evaluating the saddle point contribution to the path integral using the action

$$\frac{3k\pi^2}{\lambda\ell}\int_{\mathcal{M}} \mathrm{d}^2x\,\sqrt{g}\,\Phi^2e^{-\varphi}\left(R+\frac{2}{\ell^2}\right) + \frac{6k\pi^2}{\lambda\ell}\int_{\mathcal{M}}\sqrt{h}\,\Phi^2e^{-\varphi}(K-K_0), \tag{5.46}$$

where K is the trace of the extrinsic curvature, and $K_0 = 1$ is a constant whose role is to subtract the divergences. Note that the on-shell value of the bulk action is zero, and the total contribution is given by the GHY boundary term.

5.4.2 Black hole solution, mass and entropy from JT gravity

Let us use the previous logic to actually compute the contribution of torsion to the entropy of (2.73) black hole. Varying the action (5.42) with respect to Φ, ϕ^a, e^a, ω^{ab} and φ, we obtain the following set of equations.

$$\Phi\left(-3\lambda^{-1}e^{-\varphi}\left(R^{ab}+\ell^{-2}e^ae^b\right)+6\phi^aT^b\right)=0\,, \tag{5.47}$$

$$\begin{aligned}&\frac{-6}{\ell\lambda}e^{-\varphi}\varepsilon_{ab}R^{ab}\phi_c-\frac{6}{\ell^3\lambda}e^{-\varphi}\varepsilon_{ab}e^ae^b\phi_c-\frac{12}{\ell^3\lambda^2}\varepsilon_{ac}\left[e^{-2\varphi}\mathrm{d}\varphi e^a+e^{-2\varphi}T^a\right]-\frac{12}{\ell}\phi_c\varepsilon_{ab}e^a\mathrm{D}\phi^b\\&-\frac{6}{\ell}\varepsilon_{ac}\left[2\phi_b\mathrm{D}\phi^be^a+\phi_b\phi^bT^a\right]+\frac{12}{\ell\lambda}\varepsilon_{ac}\left[-e^{-\varphi}\mathrm{d}\varphi\mathrm{D}\phi^a+e^{-\varphi}R^{ab}\phi_b\right]-\frac{6}{\ell}\varepsilon_{ac}\Phi^2T^a=0\,,\end{aligned} \tag{5.48}$$

$$\frac{6}{\lambda\ell^3}e^{-\varphi}\left(1-\phi_c\phi^c\right)\varepsilon_{ab}e^a+\frac{6}{\ell}(1-\phi_c\phi^c)\varepsilon_{ab}\mathrm{D}\phi^a-\frac{6}{\ell^3\lambda}e^{-\varphi}\Phi^2\varepsilon_{ab}e^a=0\,, \tag{5.49}$$

$$\begin{aligned}&-\frac{1}{\ell^3\lambda^3}\varepsilon_{ab}\mathrm{D}\left[e^{-3\varphi}\Big(1+3(\lambda\ell)^2e^{2\varphi}(1-\phi_c\phi^c)\Big)\right]+\frac{6}{\ell^3\lambda^3}\Big(\lambda e^{\varphi}+\ell^2\lambda^3e^{3\varphi}(1-\phi_d\phi^d)\Big)\\&\times\frac{1}{2}\left(-\varepsilon_{ac}e^c\phi_b+\varepsilon_{bc}e^c\phi_a\right)+\frac{6}{\ell\lambda}e^{2\varphi}\left(\varepsilon_{bc}\mathrm{D}\phi^c\,\phi_a-\varepsilon_{ac}\mathrm{D}\phi^c\,\phi_b\right)+\frac{3}{\ell\lambda}\varepsilon_{ab}\mathrm{d}\left(\Phi^2e^{-\varphi}\right)=0\,,\end{aligned} \tag{5.50}$$

$$\begin{aligned}&-\frac{3}{\ell^3\lambda^3}e^{-3\varphi}\varepsilon_{ab}\left(R^{ab}+\frac{3}{\ell^2}e^ae^b\right)-\frac{1}{\ell^3\lambda^3}e^{-\varphi}\varepsilon_{ab}\left(3\ell^2\lambda^2(1-\phi_c\phi^c)\left(R^{ab}+e^ae^b\right)+6\ell^2\lambda^2\mathrm{D}\phi^a\mathrm{D}\phi^b\right)\\&\qquad-\frac{12}{\ell^3\lambda^2}e^{-2\varphi}\varepsilon_{ab}e^a\mathrm{D}\phi^b+\frac{3}{\ell\lambda}e^{-\varphi}\Phi^2\left(R^{ab}+\frac{1}{\ell^2}e^ae^b\right)=0\,.\end{aligned} \tag{5.51}$$

We set $\ell=1$ in the following. Note that equation (5.47) is analogous to the first equation in (4.88), provided that the two-dimensional torsion is zero. One solution to these equations we are interested in is precisely the dimensionally-reduced black hole we already discussed. More concretely, the solution has the line element of $1+1$ D BTZ black hole (4.99). Two-dimensional spin-connection is $\omega^{01}=r\mathrm{d}t$ and is purely determined from the vielbein. Therefore, geometry indeed satisfies the equations of JT gravity. Let us explicitly demonstrate that this BH solution satisfies equation (5.51). Note that this BH is locally AdS_2, implying $R^{ab}=e^ae^b$, so we have

$$\begin{aligned}&-6r^3\left(e^0e^1-e^1e^0\right)-6r\left(\mathrm{D}\phi^0\mathrm{D}\phi^1-\mathrm{D}\phi^1\mathrm{D}\phi^0\right)-12r^2\left(e^0\mathrm{D}\phi^1-e^1\mathrm{D}\phi^0\right)\\&=-12r^3\mathrm{d}t\mathrm{d}r-6r\left(r\sqrt{r^2-\mu}\mathrm{d}t\frac{r\mathrm{d}r}{\sqrt{r^2-\mu}}-\frac{r}{\sqrt{r^2-\mu}}r\sqrt{r^2-\mu}\mathrm{d}r\mathrm{d}t\right)\\&\qquad-12r^2\left(-\sqrt{r^2-\mu}\frac{r\mathrm{d}r}{\sqrt{r^2-\mu}}+\frac{r\mathrm{d}r}{\sqrt{r^2-\mu}}\sqrt{r^2-\mu}\mathrm{d}r\mathrm{d}t\right)=0\,,\end{aligned} \tag{5.52}$$

where we used (5.36) to derive $\mathrm{D}\phi^0=-r\sqrt{r^2-\mu}\mathrm{d}t$ and $\mathrm{D}\phi^1=-\frac{r\mathrm{d}r}{\sqrt{r^2-\mu}}$. For $\Phi=0$, the entropy of this black hole should correspond to the entropy (3.76). Using the logic of the previous subsection, we should add to this entropy the result for the entropy in JT gravity, which we computed in (4.107). Of course, we should match the parameter G_N from (4.107) to the one in (5.44). This is done by setting

$$\frac{1}{16\pi G_N} = -3k\pi^2 C^2 \,, \tag{5.53}$$

which is done by matching the prefactor of the action (5.46) and JT gravity (4.82) in Euclidean signature. Note that the constant C is also part of the JT "Newton's constant". Therefore, we should add the term $-12k\pi^3 C^2\sqrt{\mu}$ to (3.76), resulting in

$$\mathcal{S} = 4\pi^3 k\sqrt{\mu}(\mu+3) - 12k\pi^3\sqrt{\mu}C^2 = 12k\pi^3\sqrt{\mu}\left(1+\frac{\mu}{3}-C^2\right) . \tag{5.54}$$

This formula constitutes one of the main results of this chapter, and we shall discuss this formula in the following sections. Before that, we will briefly comment on the boundary terms in this two-dimensional theory.

5.4.3 Boundary terms

So far, the exact nature of the full boundary term in (5.42) has not been discussed. In this section, we will derive the boundary terms that we expect to be important for this theory. The discussion on boundary terms was conducted multiple times in this thesis, and the choice of boundary conditions in this section will be motivated by the work [31]. Similar to the case of JT gravity, we will try to remove the boundary variation of the spin-connection. Furthermore, as fields ϕ^a originate from the five-dimensional spin-connection, we will define boundary conditions for them to be such that the component $(\delta^a_b - n^a n_b)\phi^b$ "parallel" to the boundary is fixed, while the variation of the component $n^a n_b \phi^b$ should be removed from the boundary variation. Dilaton field φ is kept fixed at the boundary. The (GHY) boundary term providing us with such variation is

$$\begin{aligned} S_{GHY} = -\frac{k\pi^2}{\lambda^3 l^3}\int_{\partial\mathcal{N}} \varepsilon_{ab} e^{-3\varphi}\Big[-2\big(1+3(\lambda l)^2 e^{2\varphi}(1-\phi_c\phi^c-\Phi^2)n^a \mathrm{D}n^b\big) \\ -6\big(e^{\varphi}\lambda + e^{3\varphi}\lambda^3(1-\phi_c\phi^c-\Phi^2)\big)\phi_d n^d e^a n^b - 4\lambda^3 e^{3\varphi}(\phi_c n^c)^3 e^a n^b \\ +12 e^{2\varphi}\lambda^2\phi_c n^c n^a \mathrm{D}\phi^b - 12 e^{2\varphi}\lambda^2(\phi_c n^c)^2 n^a \mathrm{D}n^b\Big] . \end{aligned} \tag{5.55}$$

Note that the term proportional to Φ^2 in the first line of (5.55) is precisely the GHY term in first-order JT gravity (4.112).

5.5 Full five-dimenisonal theory

Let us now return to the issue of the entropy of a torsionful black hole. First, we shall derive the energy of this BH using holography, analogously to the derivation in section 3.2. First, we have to obtain the FG gauge for this BH solution. Line element in the FG gauge is given by (3.67), so that the FG gauge takes the form of

$$\begin{aligned} \overline{e}_{(0)} &= \mathrm{d}t\,, & \overline{k}^0 &= -\frac{\mu}{4}\,\mathrm{d}t\,, \\ \overline{e}^i &= \overline{e}^i_S\,, & \overline{k}^i &= \frac{\mu}{4}\,\overline{e}^i_S\,, \\ \overline{\omega}^{0i} &= 0\,, & \overline{\omega}^{ij} &= \epsilon^{ijk}\left(\overline{\omega}_{kS} - C\,\overline{e}_{kS}\right) . \end{aligned} \tag{5.56}$$

We should again try to compute the boundary energy-momentum component $\langle T^{00}\rangle$. This time, we will use a slightly different procedure, simply to complete the analysis. First, note that we have $\langle \mathcal{T}_{\alpha\beta}\rangle = e^a_\alpha\, g_{\beta\gamma}\, \langle \mathcal{T}^\gamma{}_a\rangle$, with $\langle \mathcal{T}^\gamma{}_a\rangle$ being the Hodge dual to τ_a from (3.60). From Section 3.2 it follows that

$$\langle \mathcal{T}^\alpha{}_A\rangle = \frac{k}{\sqrt{-g}}\,\epsilon^{\alpha\beta\gamma\delta}\epsilon_{abcd}\left(\frac{1}{2}\,R^{bc}_{\beta\gamma} + 2\,e^b_\beta k^c_\gamma\right) k^d_\delta\,. \tag{5.57}$$

Using the identity

$$\frac{1}{\sqrt{-g_{(0)}}}\epsilon^{tIJK}\epsilon_{0ijk}\,\tilde{e}^i_I\tilde{e}^j_J\tilde{e}^k_K = \frac{1}{\ell^3}\,\epsilon^{0ijk}\epsilon_{0ijk} = -\frac{6}{\ell^3}\,. \tag{5.58}$$

we have

$$\epsilon = \left\langle T^{00}\right\rangle = -\left\langle T^0{}_0\right\rangle = \frac{3k\mu}{2}\,\left(\frac{\mu}{2} + 1 - C^2\right)\,. \tag{5.59}$$

The total energy is again obtained by integrating the energy density over the $\mathbb{S}^3$, resulting in

$$3k\pi^2\mu\,\left(\frac{\mu}{2} + 1 - C^2\right)\,. \tag{5.60}$$

An interesting property of the last formula is that the energy is nonzero for $\mu = -1$. Note that for the torsionless solution, $\mu = -1$ in (2.73) corresponds to the global AdS spacetime. We refer to this energy as the vacuum energy on the gravity side of AdS/CFT duality

$$E_{\text{vac}} = -\frac{3k\pi^2}{2}\,. \tag{5.61}$$

Obtained energy from the boundary perspective is interpreted as the Casimir energy of a CFT on $\mathbb{R}\times\mathbb{S}^3$, as this is precisely the geometry of the dual field theory. Therefore, we have

$$E_{Casimir} = -\frac{3k\pi^2}{2}\,. \tag{5.62}$$

The total energy is negative, as expected for the CS dual [98]. If the deformation in the form of torsion is present, this energy is changed. From (5.60), we conclude

$$E_{\text{vac}}(C) = |E_{\text{vac}}(0)|(-1 + 2C^2)\,, \tag{5.63}$$

implying that the Casimir energy has a positive increase due to torsion. This modification of the Casimir energy represents yet another important deliverable from this chapter. The energy obtained in this fashion matches the energy derived in [134]. However, [134] used the Hamiltonian analysis, where the variation of the energy was calculated. In order to obtain the result (5.60), it was necessary to assume $\delta C = 0$ in the calculations. Note that for this energy, and entropy (5.54), the first law of thermodynamics $T\delta S = \delta E$ is satisfied, assuming we do not vary the constant C. Luckily, there is a way to justify this assumption. So far, we have been interested in pure CS gravity. As some part of our work was based on enhanced $SO(4,2)$ symmetry, it is a reasonable assumption to work with pure gravity, as any generic matter content would spoil this. There is, however, one option to include the matter fields that still has the desired (and even further enlarged) symmetry - the supergravity model. CS supergravities have been thoroughly studied in the past, see [33] for a review. Supergravity is a special type of gravity theory, with additional matter context, such that the local symmetry algebra is not described by the standard Lie algebra, but a more general structure called the superalgebra [135]. This is a $\mathbb{Z}_2$-graded algebra that has bosonic and fermionic generators, mixing with each other under supersymmetry transformations. The total action of the theory has both bosonic and fermionic fields, standardly coming in pairs. The action, in the bosonic sector, is taken to be of the form

$$k\int\Big[L_{\rm G}(e,\omega)+L_{\rm SU(2)}(\mathcal{A})+L_{\rm int}(A,e,\omega,\mathcal{A})\Big]\,, \tag{5.64}$$

where the first term gives the standard $5D$ AdS CS gravity (1.27) and

$$\begin{aligned} L_{\rm SU(2)} &= -\frac{\rm i}{3}\,{\rm Tr}\left(\mathcal{A}\mathcal{F}^2-\frac{1}{2}\,\mathcal{A}^3\mathcal{F}+\frac{1}{10}\,\mathcal{A}^5\right)\,,\\ L_{\rm int} &= \frac{1}{4}\left(\frac{1}{2}\,R^{ab}R_{ab}+\frac{1}{\ell^2}\,R^{ab}e_ae_b-\frac{1}{\ell^2}\,T^aT_a-\mathcal{F}^I\mathcal{F}_I\right)\,A\,. \end{aligned} \tag{5.65}$$

This theory has a BH solution with the line element given precisely by (2.73), with axial torsion (2.81).

$$\begin{aligned} e^0 &= f\,{\rm d}t\,, & e^1 &= \frac{{\rm d}r}{f}\,, & e^i &= re^i_S\,,\\ \omega^{01} &= f(f)'\,{\rm d}t\equiv\sigma{\rm d}t\,, & \omega^{0i} &= 0\,, & \omega^{1i} &= -f\,e^i_S\,,\\ \omega^i &= \tilde{\omega}^i(e)-C\,e^i_S\,, & \mathcal{A}^I &= \delta^I_i\left(\tilde{\omega}^i(e)-B\,e^i_S\right)\,, & A &= A_0{\rm d}t\,. \end{aligned} \tag{5.66}$$

The configuration of the $SU(2)$ gauge field is that of a soliton, with nontrivial Pontryagin index

$$n_1=-C\left(2-C^2\right)\in\mathbb{Z}\,. \tag{5.67}$$

As this quantity is necessarily an integer (as implied by mathematical theorems), it cannot have an arbitrarily small variation. Therefore, generically we have $\delta C=0$, resulting in the claim we previously made. Having derived this, let us now show a few more indications that the torsion contribution to the BH entropy in CS gravity is indeed given by (5.54). First, the BH entropy formula for the CS gravity with torsion was derived in [79] as a Wald entropy in the form

$$\mathcal{S}_{\rm Wald}=2\pi k\int\limits_{r_h}\varepsilon_{abcde}\,n^{ab}\left(R^{cd}e^e+\frac{1}{3\ell^2}\,e^ce^de^e\right)\,, \tag{5.68}$$

where n^{ab} is biorthogonal, whose only nonzero components are given as $n^{01}=-n^{10}=\frac{1}{2}$. This result is essentially the same as the standard Wald's entropy for the Riemannian geometry, with the difference that the R^{ab} in (5.68) is the Riemann-Cartan curvature, having a contribution from the contorsion field. Computing this quantity for the BH solution (2.73) with torsion, we have

$$\begin{aligned} \mathcal{S}_{\rm Wald} &= 4k\pi\int\limits_{r_h}\varepsilon_{01ijk}n^{01}\left(R^{ij}e^k+\frac{1}{3}e^ie^je^k\right)\\ &= 2k\pi\int\limits_{r_h}\varepsilon_{ijk}\left(R^{ij}e^k+\frac{1}{3}e^ie^je^k\right)\\ &= 2k\pi\int\limits_{r_h}\varepsilon_{ijk}e^ie^je^k\left(r(1-C^2-f^2)+\frac{r^3}{3\ell^2}\right)\\ &= 2k\pi\int\limits_{r_h}{\rm d}\Sigma\;3!\left(r_h(1-C^2)+\frac{r_h^3}{3\ell^2}\right)\\ &= 24\pi^3k\left(\frac{\sqrt{\mu}^3}{3}+\sqrt{\mu}(1-C^2)\right)\,, \end{aligned} \tag{5.69}$$

which coincides with (5.54) up to an irrelevant proportionality constant. At this stage, we shall again emphasize that the result (5.54) can be obtained by using precisely the same Wald's entropy formula as in the standard Riemannian case, but with Riemann-Cartan curvature. As Wald's entropy is closely related to the entanglement entropy in the RT, we conjecture that the correct expression for the entanglement entropy for a theory dual to the CS gravity is given by the same expression as in the Riemannian case, but with Rieman-Cartan quantities used instead of the Riemannian ones [136]. More concretely, we conjecture that the entanglement entropy in the QFT dual to CS gravity, assuming the bulk geometry is given by the AdS spacetime with torsion ((2.76) with $\mu = 0$) contains the divergent term of the form $\mathcal{A}C^2 \ln \varepsilon$, with $\mathcal{A}$ being the entangling surface of the analysed boundary region.

The second approach we shall use is the one developed in [74]. We compute the energy as the boundary charge related to the Killing vector field $\xi = \partial_t$, while the entropy is obtained from the contribution integrated over the BH horizon. This is perfectly in the spirit of Wald's approach to the BH entropy. The starting point is the definition of covariant momenta

$$\Pi_{ab} = \frac{\partial L}{\partial R^{ab}} \,, \qquad \Pi_a = \frac{\partial L}{\partial T^a} \,, \qquad \Pi = \frac{\partial L}{\partial F} = 0 \,. \tag{5.70}$$

Computing these quantities for the action (5.64), we obtain.

$$\begin{aligned} \Pi_{ab} &= \frac{k}{2}\,\varepsilon_{abcde}\left(R^{cd}e^e + \frac{1}{3}\,e^c e^d e^e\right) + \frac{k}{2}\left(R_{ab} + e_a e_b\right)A \,, \\ \Pi_a &= -\frac{k}{2}\,T_a A \,. \end{aligned} \tag{5.71}$$

The next step is to focus on the variation

$$\delta\mathcal{G} = \iota_\xi e^a \delta\Pi_a + \delta e^a \iota_\xi \Pi_a + \frac{1}{2}\iota_\xi \omega^{ab}\delta\Pi_{ab} + \frac{1}{2}\delta\omega^{ab}\iota_\xi \Pi_{ab} + \iota_\xi A \delta\Pi + \delta A \iota_\xi \Pi \,. \tag{5.72}$$

In this formula, contraction ι_ξ is done with respect to the Killing vector field $\xi = \partial_t$. This variation shall be computed at the asymptotic infinity and at the horizon. At infinity, we vary the parameter μ, while at the horizon, the variation is performed keeping the surface gravity fixed. The first law of thermodynamics is expressed as

$$\delta\Gamma(\infty) = \delta\Gamma(r_h) \,, \tag{5.73}$$

where $\delta\Gamma(r_h) = \int_{r_h} \delta\mathcal{G} = T\delta S$. A concrete computation gives

$$\begin{aligned} \Pi_{01} &= \frac{12ke_S^2 e_S^3 e_S^4}{4}\left[r(1 - C^2 - f^2) + \frac{r^3}{3}\right] \,, \\ \Pi_{1i} &= \frac{k}{8}\left[-4rf\sigma + 2\left(1 - C^2 - f^2\right)f - \frac{fC}{2}\,A_0 + 2r^2 f\right]\epsilon_{ijk}\mathrm{d}te^j e^k \,, \\ \Pi_{ij} &= k\left[-f^2\sigma + \frac{1}{4}\left(1 - C^2 - f^2 + r^2\right)\right]\mathrm{d}te_i ee_j \,. \end{aligned} \tag{5.74}$$

Furthermore, we have

$$\begin{aligned} \Pi_0 &= 0 \,, \\ \Pi_1 &= 0 \,, \\ \Pi_i &= -\frac{k}{2} rC\varepsilon_{ijk}e^j e^k A_0 \mathrm{d}t \,. \end{aligned} \tag{5.75}$$

It is then easy to obtain that the variation at infinity is given as

$$\delta\mathcal{G} = 6ke_S^2e_S^3e_S^4\delta\mu\left(\frac{r^2}{2}+\frac{1}{2}\left(\frac{1}{2}\left(-\frac{r^2}{2}+(1-C^2-f^2)\right)-\frac{CA_0}{2}\right)\right)$$
$$= 6ke_S^2e_S^3e_S^4\delta\mu\left(\frac{1}{4}\left(1-C^2+\mu\right)-\frac{CA_0}{4}\right) \tag{5.76}$$

Variation at the horizon is performed using the fact that the surface gravity is kept constant. For example, we have

$$\int_{r_h}\iota_\xi\omega^{01}\delta\Pi_{01} = \frac{kr_h}{2}\delta\left(\left((1-C^2)+\frac{r_h^2}{3}\right)r_h\right)\int_{r_h}\varepsilon_{ijk}e^ie^je^k \tag{5.77}$$
$$= 6ke_S^2e_S^3e_S^4\sqrt{\mu}\delta\left((1-C^2)r_h+\frac{r_h^3}{3}\right). \tag{5.78}$$

Disregarding the gauge field, which is not of interest in the pure CS gravity, the end result again coincides with (5.60) and (5.54). Finally, let us mention two related works that are very relevant for this thesis. First, one direction in which the previous analysis should be continued is the addition of other, more general matter fields in the bulk. One way to do this is to include them directly in the two-dimensional model, as it turns out that, in this case, equations of motion can often be solved analytically (assuming we are interested in a stationary spacetime). This line of work is conducted in collaboration with Stefan Djordjevic. Further, a more systematic approach to the BH entropy can be done, relying on the minisuperspace approximation, as in [85]. One first chooses a set of fields depending only on one variable (in this case, on r) and performs a procedure similar to the dimensional reduction, respecting the symmetry. Once a consistent truncation is found, it is much simpler to perform the analysis of the thermodynamic properties of the BH solution.

Chapter 6

Gauge fields on Riemann-Cartan background and holography

This chapter is based on author's work [137].

Apart from initial pedagogical considerations, most of the work in this thesis was carried out in the regime of pure gravity, with the exception of the SUGRA solution that was discussed in Chapter 5.3, and partial comments made in the $2D$ reduced theory. Matter fields are, of course, important, and a natural question is whether we can incorporate them in our study. Having this question in mind, here we study the dynamics of a gauge field in the BH geometry with torsion, and apply the holographic dictionary to understand the boundary dual of this theory. In order to get new and interesting physics, we shall use a nonminimal coupling between the gauge field and the torsion. This is not surprising, as we already commented that the nonminimal coupling is necessary in CS gravity to obtain novel solutions to the equations of motion. However, here we will not be solving gravitational equations. Our work [137] is, as far as we are aware, the first attempt to merge the holographic analysis of matter fields and the Riemann-Cartan geometry in the context of condensed-matter physics. For this reason, we believe it is justified to simplify our analysis and work in the probe limit, where the matter dynamics is decoupled from the gravitational action, leaving the inclusion of the backreaction for future work.

6.1 Minimal vs non-minimal coupling

Assuming we work in the probe limit, minimal coupling of $U(1)$ gauge field to the spacetime geometry is blind to torsion. This follows from the fact that the YM action is given by

$$-\frac{1}{2e^2}\int \mathrm{d}A \wedge \star \mathrm{d}A\,, \tag{6.1}$$

which does not depend on the affine structure of spacetime. Here, we used the standard definition of gauge curvature $F = \mathrm{d}A$, or in components $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In Riemannian spaces, we have an equivalent expression $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, which could perhaps motivate that on Riemann-Cartan spaces we should use the same expression as the definition of the gauge curvature in the YM action (6.1). However, we have

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + T^\rho_{\mu\nu} A_\rho\,. \tag{6.2}$$

This is not a gauge invariant object, and under $A_\mu \to A_\mu + \partial_\mu \lambda$ it transforms as

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu \to \nabla_\mu A_\nu - \nabla_\nu A_\mu + T^\rho_{\mu\nu} \partial_\rho \lambda \,. \tag{6.3}$$

Only if the term $T^\rho_{\mu\nu}\partial_\rho\lambda$ vanishes in the action (which is, of course, not true generally), this can be a suitable definition for the curvature used in defining a gauge invariant theory. For this reason, we shall adopt the (6.1) action, which yields the theory unaffected by the presence of the torsion field. To incorporate the effects of torsion, we will introduce a nonminimal coupling between the gauge field and the torsion.

Even though we are not aware of any other work, apart from our own, trying to incorporate torsion and EM fields in a holographic set-up, the interplay between torsion and EM fields is well-explored in the literature. Perhaps the main motivation comes from the cosmological considerations, where the torsion may play a certain role. Motivated by the work [138], we will take the following nonminimal coupling between the gauge field and torsion, controlled by the coupling we call g

$$-\frac{1}{2e^2} \int F \wedge (1 - g^2 \star (T^a \wedge \star T_a)) \star F \,. \tag{6.4}$$

Actually, this form of coupling is only briefly mentioned in [138] because this paper, for phenomenological reasons, addresses the issue of four-dimensional spacetime. In this case, it is possible to consider the coupling of the form

$$-\frac{1}{2e^2} \int F \wedge (1 - g^2 \star (T^a \wedge T_a)) \star F \,, \tag{6.5}$$

because $T^A \wedge T_A$ is a top form. However, in our case, we have to use the combination (6.4). Furthermore, our signature differs from the one used in [138], so there are sign differences in our formulas.

More general couplings between $U(1)$ gauge field and torsion, that are at most quadratic in torsion, have been analyzed in [139]. An appealing alternative to coupling (6.4) is given by the Preuss term

$$\int (T^a \wedge F) \wedge \star (T_a \wedge F) \,, \tag{6.6}$$

which was introduced in [56] to analyse astrophysical implications of torsion. Through this chapter, we will commit to the use of (6.4), though we will later comment on the possibility of including term (6.6). As we will see, no conceptual difference arises in our conclusions.

6.2 Spin-current of the background

For the bulk soluton we again consider (2.76) with axial torsion. A simple calculation gives

$$1 - g^2 \star (T^a \wedge \star T_a) = 1 + \frac{3g^2C^2}{r^2} \equiv \mathcal{C} \,. \tag{6.7}$$

The time component of the boundary torsion is zero

$$\overline{T}^0 = \mathrm{d}\overline{e}^0 + \overline{\omega}^0{}_i \wedge \overline{e}^i = 0 \,, \tag{6.8}$$

while spatial components are given by

$$\overline{T}^i = C\varepsilon_{ijk} \mathrm{d}x^j \wedge \mathrm{d}x^k \,. \tag{6.9}$$

Boundary geometry in this case coincides with the Minkowski spacetime, but endowed with the nontrivial torsion (6.9). We believe that this is one of the simplest possible forms of a spacetime on which the effect of torsion can be studied. Using relation (3.66) from Chapter 3, we have

$$\langle \mathcal{S}_{ij} \rangle_{\mathrm{QFT}} = k\mu\delta \, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}t \,. \tag{6.10}$$

Taking the Hodge dual of the last expression, we have

$$\star \langle \mathcal{S}_{ij} \rangle_{\mathrm{QFT}} = k\mu\delta\varepsilon_{ijk}\mathrm{d}x_k . \tag{6.11}$$

Interestingly, the spin-current is proportional to the square of the temperature at the boundary (2.68). From the spin-current expression (6.11) one can compute the axial current, as advocated in [25, 87]. The idea behind this is that if we consider a four-dimensional massless Dirac fermion with the standard action

$$\int \mathrm{d}^4x \ |e| \, \overline{\psi}\gamma^\mu \left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} \right) \psi \,, \tag{6.12}$$

variation of this action with respect to the spin-connection gives a spin-current such that the combination

$$\mathcal{J}^\mu_{\mathrm{ax}} = \frac{1}{6}\varepsilon^{abcd} e^\mu_a \sigma_{bcd} \tag{6.13}$$

is proportional to $\overline{\psi}\gamma^5\gamma^\mu\psi$, which is the axial current. Generalizing this idea, one can define the axial current to be given by the expression (6.13) for a generic spin-current. Taking the example from this section, we obtain

$$\mathcal{J}^0_{\mathrm{ax}} = k\mu\delta \,, \qquad \mathcal{J}^i_{\mathrm{ax}} = 0 \,. \tag{6.14}$$

6.3 Equations of motion and holographic renormalization

Equations of motion follow directly from (6.4) and are given by

$$\mathrm{d} \left(\mathcal{C} \star F \right) = 0 \,, \tag{6.15}$$

or, writing down in a local coordinate basis,

$$\partial_\mu \left(\sqrt{-g}\mathcal{C}F^{\mu\nu} \right) = 0 \,. \tag{6.16}$$

Decomposing a free index ν as $\nu = \alpha$ and $\nu = 1$, we get two equations

$$\partial_\alpha F^{\alpha r} = 0, \tag{6.17}$$

$$\partial_r \left(\sqrt{-g}\mathcal{C}\frac{(r^2-\mu)}{r^2}\partial_r A_\alpha \right) + \sqrt{-g}\mathcal{C}\partial_\beta F^{\beta\alpha} = 0. \tag{6.18}$$

As this is a gauge theory, we have redundancies and the freedom to choose a gauge condition (as long as this does not affect any physical charges). We take the gauge $A_r = 0$. The equation (6.18) then gives the asymptotic expansion of the boundary fields around $r = +\infty$ as

$$A_\alpha \sim A^{(0)}_\alpha + B_\alpha \frac{\ln r}{r^2} + \frac{A^{(1)}_\alpha}{r^2} \,, \tag{6.19}$$

where $B_\alpha = \frac{1}{2}\partial_\beta f^{\beta\alpha}$ for $f_{\alpha\beta} = \partial_\alpha A^{(0)}_\beta - \partial_\beta A^{(0)}_\alpha$ being the curvature for the background boundary gauge field $A^{(0)}$. From now on in this chapter, we take the convention that boundary indices are raised and lowered using the flat Minkowski metric. A holographic analysis of this system closely follows the case of minimal coupling, which can be found in the literature, see for example [140]. For completeness, we will explicitly show it here, relying on Chapter 3 for general ideas. Starting from (6.4), we first perform a partial integration, with the bulk term on-shell vanishing, to get

$$\int_{\mathcal{N}} \mathrm{d}\left(\left(1-g^2\star(T^A\star T_A)\right)A\star F\right)=\int_{\partial\mathcal{N}}\left(1-g^2\star(T^A\star T_A)\right)A\star F\,. \tag{6.20}$$

After plugin in the form (6.19) into (6.20), we get

$$\int_{\partial\mathcal{N}}\left(1+\frac{3\delta^2C^2}{r^2}\right)\left(A^{(0)}+B\frac{\ln r}{r^2}+\frac{A^{(1)}}{r^2}\right)\left(\star\left(\left(B\left(\frac{1}{r^3}-\frac{2\ln r}{r^3}\right)-2\frac{A^{(1)}}{r^3}\right)\mathrm{d}r\right)\right)\,, \tag{6.21}$$

where $\star$ is the bulk Hodge dual, and we disregarded all boundary terms proportional to $\mathrm{d}r$, as the boundary is defined for $r=R=\mathrm{const}$, in the limit $R\equiv\frac{1}{\varepsilon}\to+\infty$. Focusing only on terms that are either finite or infinite in the limit $\varepsilon\to0$, we obtain

$$S_{\mathrm{on-shell}}=-\frac{1}{2e^2}\int_{\partial\mathcal{N}}\mathrm{d}^4y\;A^{(0)}_\alpha B^\alpha\left(1-2\ln R\right)-2A^{(0)}_\alpha A^{(1)\alpha}\,. \tag{6.22}$$

This action is infinite due to the $\ln R$ term, and appropriate holographic renormalization has to be performed. Similarly, and even more in the spirit of Chapter 3, we compute the on-shell variation of the action

$$\delta S_{\mathrm{on-shell}}=-\frac{1}{e^2}\int_{\partial\mathcal{N}}\mathrm{d}^4y\;\delta A^{(0)}_\alpha\left((1-2\ln R)B^\alpha-2A^{(1)\alpha}\right)\,. \tag{6.23}$$

The variation is also infinite, but this infinity can be removed by adding the boundary counterterm

$$S_{ct}=-\frac{1}{e^2}\int\mathrm{d}^4y\;\ln R\,A^{(0)}_\alpha B^\alpha\,. \tag{6.24}$$

Note that the reason why this works lies in the fact that B_α is secretly the second derivative of $A^{(0)}$, so the variation of B_α can be, after two partial integrations, written as the variation of $A^{(0)}$. Focusing on the finite part, we obtain (we put $e=1$)

$$\langle J^\alpha\rangle_{QFT}=\frac{\delta S_{\mathrm{ren}}}{\delta A^{(0)}_\alpha}=2A^{(1)\alpha}-B^\alpha=2A^{(1)\alpha}-\frac{1}{2}\partial_\beta f^{\beta\alpha}\,. \tag{6.25}$$

At this stage, we have the freedom to choose any finite counterterm that does not spoil the Dirichlet boundary conditions $\delta A^{(0)}=0$ at the boundary. Two different possibilities were discussed in [140]. For example, we could add the counterterm

$$-\frac{1}{4}\int\mathrm{d}^4y\left(\frac{1}{2}-\ln R\right)f_{\alpha\beta}f^{\alpha\beta}\,, \tag{6.26}$$

which would correspond to a different choice of a finite counterterm than the one we made. Another possibility is to add the counterterm

$$-\frac{1}{4}\int\mathrm{d}^4y\sqrt{-h}\left(\frac{1}{2}-\ln R\right)F_{\alpha\beta}F^{\alpha\beta}\,, \tag{6.27}$$

which would give only the first term in (6.25).

6.4 Holographic conductivity

Bulk gauge field is dual to boundary conserved $U(1)$ current. The fact that the current is conserved follows from (6.17). One way to extract boundary optical conductivity is to couple the boundary theory with a source term of the form $A^{(0)} \sim \text{const} \cdot e^{i\omega t}$ and take the ratio of the current one-point function and the electric field $E_i = -\partial_t A_i^{(0)}$, yielding

$$\sigma(\omega) = \frac{\langle J_x \rangle}{E_x^0} = \frac{2A_x^{(1)} + \frac{1}{2}\partial_t^2 A_x^0}{-\partial_t A_x^{(0)}} = \frac{2}{i\omega}\frac{A^{(1)}}{A^{(0)}} + i\frac{\omega}{2}\,. \tag{6.28}$$

Note that, using the theory of linear response, conductivity can be obtained from the current-current two-point function, giving essentially the same result (see also [141, 142, 143]). As our goal is to probe the conductivity in the x direction, we make an ansatz for the bulk field as $A_x = A_x(z)e^{-i\omega t}$. Equation (6.18) then takes the form

$$\frac{\mathrm{d}^2 A_x}{\mathrm{d}r^2} + \frac{3g^2C^2\mu + 3r^4 + r^2\left(3g^2C^2 - \mu\right)}{r\left(r^2-\mu\right)\left(3g^2C^2 + r^2\right)}\frac{\mathrm{d}A_x}{\mathrm{d}r} + \frac{\omega^2}{(r^2-\mu)^2}A_x = 0\,. \tag{6.29}$$

In this section, we will be interested in computing three quantities. First, we compute the AC conductivity when $C = 0$, which is the Riemannian case. Next, we compute the DC conductivity in the general case of $C \neq 0$, which is an analytically tractable calculation. Finally, we compute the full AC conductivity for $C \neq 0$, relying on the numerical shooting method.

6.4.1 $C = 0$ AC conductivity

In the Riemannian case, the equation (6.29) simplifies to

$$\frac{\mathrm{d}^2 A_x}{\mathrm{d}r^2} + \frac{3r^2 - \mu}{r\left(r^2-\mu\right)}\frac{\mathrm{d}A_x}{\mathrm{d}r} + \frac{\omega^2}{(r^2-\mu)^2}A_x = 0\,. \tag{6.30}$$

Further, we make the following rescaling $r = t\sqrt{\mu}$ to get

$$\frac{\mathrm{d}^2 A_x}{\mathrm{d}t^2} + \frac{3t^2 - 1}{t\left(t^2-1\right)}\frac{\mathrm{d}A_x}{\mathrm{d}t} + \frac{\omega^2}{\mu(t^2-1)^2}A_x = 0\,. \tag{6.31}$$

This equation has analytic solutions. It is a second-order linear differential equation and thus has two independent solutions. Even though in the bulk of this thesis it was irrelevant whether we use the GPKW dictionary in Euclidean or Lorentzian signature, here, in order to compute holographic conductivity, we have to be more careful. As standard in similar computations [144], we should impose ingoing boundary conditions at the BH horizon. This is, of course, only possible in the Lorentzian signature. To explicitly see how this boundary condition is implemented, let us write the equation (6.31) assuming $t \approx 1$, so that the near-horizon limit is implemented. Performing a simple change of variable as $\delta t = t - 1$, we have

$$\frac{\mathrm{d}^2 A_x}{\mathrm{d}\,\delta t^2} + \frac{1}{\delta t}\frac{\mathrm{d}A_x}{\mathrm{d}\,\delta t} + \frac{\omega^2}{4\mu\,\delta t^2}A_x = 0\,. \tag{6.32}$$

This equation can be solved by taking $A_x \sim \delta t^\alpha$, resulting in

$$\alpha(\alpha-1) + \alpha + \frac{\omega^2}{4\mu} = 0\,, \tag{6.33}$$

whose two solutions are $\alpha = \pm i\frac{\omega}{2\mu}$. The ingoing solution then takes the asymptoti form $\sim \delta t^{\frac{-i\omega}{2\sqrt{\mu}}} = (t-1)^{\frac{-i\omega}{2\sqrt{\mu}}}$. With this, we are ready to write the solution to (6.31) with ingoing boundary conditions at the horizon. It is given by

$$(t-1)^{\frac{-i\omega}{2\sqrt{\mu}}}(t+1)^{\frac{-i\omega}{2\sqrt{\mu}}}\,{}_2F_1\left(1-\frac{i\omega}{2\sqrt{\mu}},\frac{-i\omega}{2\sqrt{\mu}};\frac{-i\omega}{\sqrt{\mu}}+1;1-t^2\right), \tag{6.34}$$

where we do not care about the overall normalization, as it is not important for computing the holographic conductivity. As the equation for the gauge field A_x is analytically solvable in this case, we simply have to expand the solution around the boundary $t \to +\infty$ and extract the AC conductivity using (6.28). This can efficiently be done using Wolfram Mathematica and working in variable $\frac{1}{t}$, resulting in

$$\sigma(\omega) = i\omega\left(\psi\left(1-\frac{i\omega}{2\sqrt{\mu}}\right)+\gamma\right)+\sqrt{\mu}, \tag{6.35}$$

with $\psi(x) = \frac{\mathrm{d}\ln\Gamma(x)}{\mathrm{d}x}$ being the digamma function. For the later use, we plot the real and the imaginary part of (6.35) in figures 6.1 and 6.2.

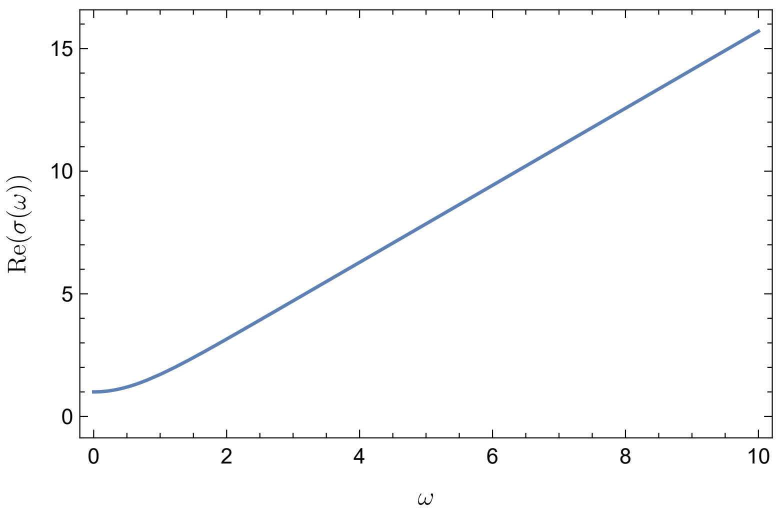

Figure 6.1: Real part of holographic conductivity for the Riemannian geometry; $\mu = 1$.

6.4.2 Direct current conductivity

Additionally, it is well-known that one can often compute the DC conductivity without having to fully solve the equation of motion for the gauge field [93]. In general, one could start from the equation of motion for general C (6.29) and take the $\omega \to 0$ limit in the solution, but this is often tricky and requires a full solution that we do not have in an analytic form. Luckily, there is a trick we can use to extract only the DC conductivity. Before we proceed, we note that it is easier to work with the variable $\mathsf{z} = \frac{1}{r}$, so that the equation (6.29) now takes the form

$$\frac{\mathrm{d}^2A_x}{\mathrm{d}\mathsf{z}^2}+\frac{3-\frac{2}{3g^2C^2\mathsf{z}^2+1}+\frac{2}{\mu\mathsf{z}^2-1}}{\mathsf{z}}\frac{\mathrm{d}A_x}{\mathrm{d}\mathsf{z}}+\frac{\omega^2}{(1-\mathsf{z}^2\mu)^2}A_x = 0. \tag{6.36}$$

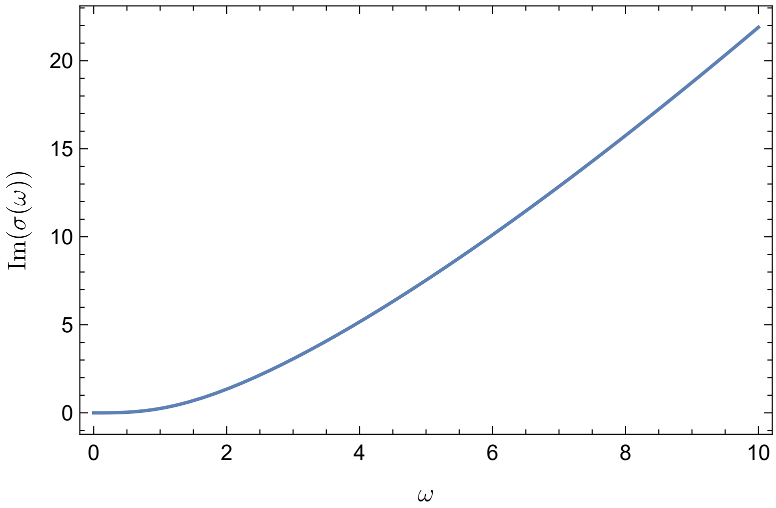

Figure 6.2: Imaginary part of holographic conductivity for the Riemannian geometry; $\mu = 1$.

The trick is to use the ansatz for the gauge field in the form $\delta A = (-Et + a(\mathtt{z}))\mathrm{d}x$, so that we get

$$\partial \mathtt{z} \left(\left(1 + 3g^2C^2\mathtt{z}^2\right) \left(1/\mathtt{z} - \mu\mathtt{z}\right) \partial_{\mathtt{z}} a(\mathtt{z}) \right) = 0 \,. \tag{6.37}$$

This equation shows that the quantity $\left(1 + 12g^2C^2\mathtt{z}^2\right) \left(1/\mathtt{z} - \mu\mathtt{z}\right) \partial_{\mathtt{z}} a(\mathtt{z})$ is conserved when moving between different $\mathtt{z} = \text{const}$ slices. Near the $\mathtt{z} \to 0$ boundary, we have

$$\left(1 + 3g^2C^2\mathtt{z}^2\right) \left(1/\mathtt{z} - \mu\mathtt{z}\right) \partial_{\mathtt{z}} a(\mathtt{z}) \to \frac{1}{\mathtt{z}} \partial_{\mathtt{z}} a(\mathtt{z}) = 2A^{(1)} \,. \tag{6.38}$$

Next, we should impose the boundary conditions at the horizon. As $\omega = 0$, we cannot use the same procedure as before, and instead we shall insist that the quantity $F_{\mu\nu}F^{\mu\nu}$ is finite at the horizon. This regularity condition enforces the relation

$$\partial_{\mathtt{z}} a(\mathtt{z})|_{\mathtt{z}_h} = \frac{E}{1 - \mu\mathtt{z}^2}|_{\mathtt{z}_h} \,, \tag{6.39}$$

and this, in turn, enables us to compute the conserved quantity at the horizon. Equating the value of the conserved quantities at the horizon and the asymptotic boundary, we have

$$2A^{(1)} = \left(1 + \frac{3g^2C^2}{\mu}\right) \sqrt{\mu} E \,. \tag{6.40}$$

Using the relation $\sigma_{DC} = \frac{\langle J_x \rangle}{E}$, we get

$$\sigma_{DC} = \sqrt{\mu} + \frac{3g^2C^2}{\sqrt{\mu}} \,. \tag{6.41}$$

The inverse of conductivity (the resistivity $\rho = \frac{1}{\sigma}$) is plotted against the temperature (2.68) in figure 6.3. At low temperatures, resistivity is linear in temperature, a consequence of our non-minimal coupling to torsion (6.4). Furthermore, there is a transition temperature at $T_c = \frac{\sqrt{3}gC}{2\pi}$, above which the resistivity is a decreasing function of temperature. Despite this nice behaviour (resembling both metallic and semiconductor behaviour), we have to stress that our calculation was done in the probe limit, and it is known that in this limit, DC conductivity should not be taken completely seriously. For example, even in the most famous consideration of holographic conductivity ([22] and references therein), we can see that DC conductivity, obtained as the limit $\omega \to 0$ of the real part of AC conductivity, tends to some finite value, even though in that model it is expected that DC conductivity diverges. Nevertheless, our computation of the DC conductivity in this subsection will serve as a sanity check for the numerical computation that follows. Finally, note that in our work, BH in the bulk is not electrically charged, and thus we do not expect infinite conductivity at the boundary.

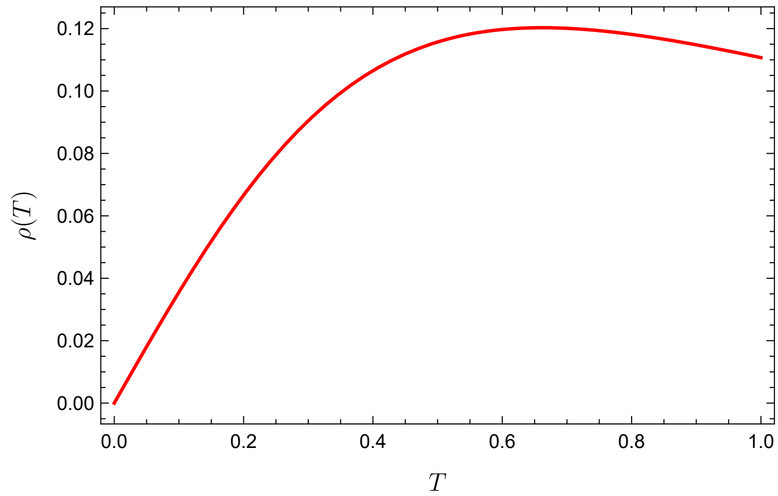

Figure 6.3: DC conductivity in terms of temperature; C and g are held fixed.

6.4.3 Numerical analysis for $C \neq 0$

Unfortunately, the torsionful case does not have an analytic solution for the AC conductivity. Therefore, we use numerical methods, implemented in Wolfram Mathematica. We will not present the whole code, as this is a very standard procedure, and the code used in this work was obtained by modifying the preexisting code by Mihailo Cubrovic[1]. We start by solving the equation near the horizon.

```
f[z_] = 1/z - z*\[Mu];
EOM = Evaluate[
   D[(1 + 3*(gC)^2*z^2)*f[z] D[Ax[z], z],
     z] + (omega)^2*(1 + 3*(gC)^2*z^2)/
     (z^2 f[z]) Ax[z]];
Ax[z_] = (1/Sqrt[\[Mu]] - z)^(-I*omega/AA) Sum[
   s[i] (\[Mu] - z)^(i - 1), {i, 1, 1}]
EOMLeading = Simplify[(1/Sqrt[\[Mu]] - z)^(I*omega/AA) EOM]
```

Further terms in the expansion are obtained by expanding the field in a power series near the horizon and solving algebraic relations for the series coefficients. The code implementing this is

```
    HorOrder = 6;
Ax[z_] = (1/Sqrt[\[Mu]] - z)^(-I*omega/(AA)) (1 +
     Sum[s[i] (1/Sqrt[\[Mu]] - z)^(i), {i, 1, HorOrder}]);
EOMHor = Simplify[(1/Sqrt[\[Mu]] - z)^(I*omega/(AA)) EOM];
```

[1]To whom the authors express a deep gratitude.

```
EOMHorSer =
  s*Normal[Series[EOMHor, {z, 1/Sqrt[\[Mu]], HorOrder}]] /. {z ->
     1/Sqrt[\[Mu]] - s};
Cons = Table[Coefficient[EOMHorSer, s^i], {i, 1, HorOrder}];
HorSolCoeff = Solve[Cons == 0, Table[s[i], {i, 1, HorOrder}]]
FullHorSol[
   z_] = (1/Sqrt[\[Mu]] - z)^(-I*omega/(AA)) (1 +
      Sum[s[i] (1/Sqrt[\[Mu]] - z)^(i), {i, 1, HorOrder}]) /.
   HorSolCoeff[[1]];
```

The idea is then to use MATHEMATICA's built-in function NDSOLVE to numerically get the solution of the equation

```
    zb = 10^(-6);
zHor = 1/Sqrt[\[Mu]] - zb;
BCHor1 = FullHorSol[zHor];
BCHor2 = D[FullHorSol[z], z] /. {z -> zHor} // Simplify;
AxAns[zVal_, omeVal_] :=
  Ax[zVal] /.
   NDSolve[Evaluate[{EOM == 0, BCHor1 == Ax[zHor],
       BCHor2 == Ax'[zHor]} /. {omega -> omeVal}], Ax,
       {z, zb, zHor}];
```

After obtaining the solution for A_x component, we should use the asymptotic expansion (6.19) to obtain the one-point function, and then use (6.28) to extract the holographic AC conductivity, which is effectively done by the command

```
    sigSol = (-I/(omeVal)) (D[D[sigFunc[z], z], z]/sigFunc[z] +
       omeVal^2/2*(3 + 2*Log[z])) + (I*omeVal/2) /. {z -> zb};
```

The relevant graphs for the real and the imaginary part of $\sigma(\omega)$ are given in figures 6.4 and 6.5.

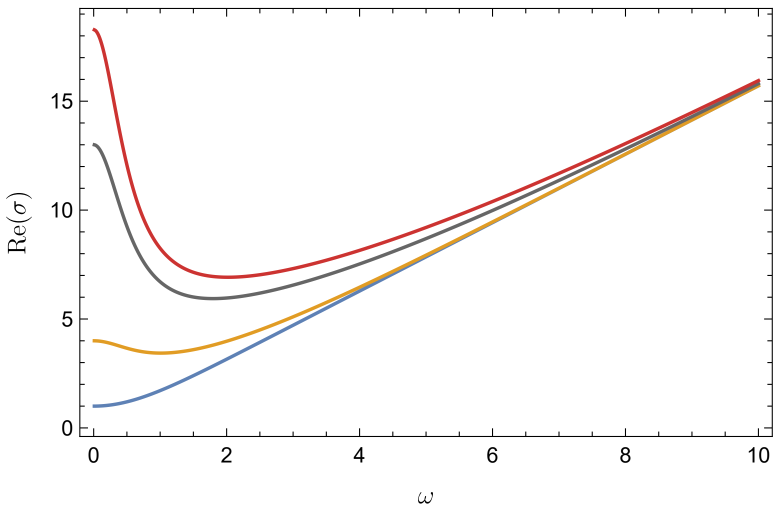

Figure 6.4: Holographic AC conductivity (real part) for different values of torsion constant C ($gC = 0,\ 1,\ 2,\ 2.4$ respectively from bottom to top); $\mu = 1$.

First, note that the bottom (blue) line in both figures corresponds to the $C = 0$ case, where we should reproduce the results of subsection 6.4.1. Indeed, it is evident that the bottom line,

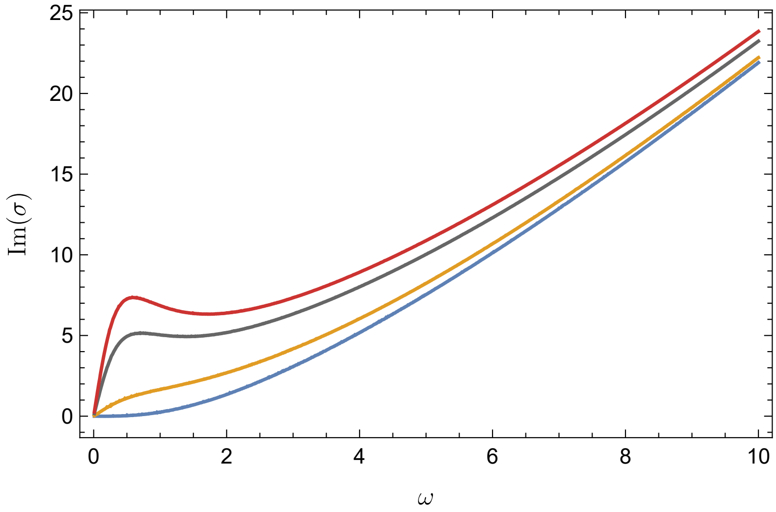

Figure 6.5: Holographic AC conductivity (imaginary part) for different values of torsion constant C ($gC = 0,\ 1,\ 2,\ 2.4$ respectively from bottom to top); $\mu = 1$.

obtained numerically in this section, matches the analytical result from subsection 6.4.1, given in figures 6.1 and 6.2. Furthermore, the intersection of lines with the vertical $\omega = 0$ axis should reproduce the results for DC conductivity. Again, it is easy to see that this is indeed the case. Finally, note that even for $C \neq 0$ we have a linear in ω behaviour for large frequencies, as expected for a five-dimensional bulk.

Before proceeding to the discussion of our results, we briefly comment on an alternative coupling (6.6). In this case, the analogue of equation (6.29) is

$$\frac{\mathrm{d}^2 A_x}{\mathrm{d}r^2} + \frac{2g^2C^2\mu + 3r^4 + r^2\left(2g^2C^2 - \mu\right)}{r\left(r^2-\mu\right)\left(2g^2C^2 + r^2\right)}\frac{\mathrm{d}A_x}{\mathrm{d}r} + \frac{\omega^2}{\left(r^2-\mu\right)^2}A_x = 0\,, \tag{6.42}$$

which essentially takes the same form as (6.29), so that no crucial difference arises in a subsequent analysis of the holographic conductivity. From the last equation, one might conjecture that the only difference between couplings (6.4) and (6.6) is the change $3g^2C^2 \to 2g^2C^2$, but this is not entirely correct. For example, if we were to introduce a finite chemical potential at the boundary, we need to solve the equation for A_t component of the gauge field. By making the ansatz $A = A_t(r)\mathrm{d}r$, we have

$$r\left(3g^2C^2 + r^2\right)\frac{\mathrm{d}^2 A_t}{\mathrm{d}r^2} + 3\left(g^2C^2 + r^2\right)\frac{\mathrm{d}A_t}{\mathrm{d}r} = 0\,, \tag{6.43}$$

which is solved by

$$A_t(r) = \mu - 2\rho\left(\frac{\ln\left(1 + \frac{3g^2C^2}{r^2}\right)}{6g^2C^2}\right)\,. \tag{6.44}$$

In this expression, μ and ρ are integration constants, representing the boundary chemical constant and charge density. On the other hand, for the alternative coupling (6.6), we have the equation

$$r\left(6g^2C^2 + r^2\right)\frac{\mathrm{d}^2 A_t}{\mathrm{d}r^2} + 3\left(2g^2C^2 + r^2\right)\frac{\mathrm{d}A_t}{\mathrm{d}r} = 0\,, \tag{6.45}$$

with solution

$$A_t(r) = \mu - 2\rho\left(\frac{\ln\left(1 + \frac{6g^2C^2}{r^2}\right)}{12g^2C^2}\right)\,. \tag{6.46}$$

Here, it is obvious that the previous substitution $3g^2C^2 \to 2g^2C^2$ does not hold. Nevertheless, the form of the solutions is similar, so we do not expect any different behaviour in the boundary duals for (6.4) and (6.6) couplings.

6.5 Interpretation of the results

Let us recall the simplest set-up for calculating real and imaginary optical conductivity of a metal - the Drude model (see any textbook reference in basic condensed matter physics, [145]). This model uses classical mechanics of electrons of mass m in a metal, with a stochastic damping force, resulting in equation

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = -e\vec{E} - \frac{\vec{p}}{\tau}\,, \tag{6.47}$$

where $p = m\vec{v}$ is the momentum of an electron. We are interested in $\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = 0$ and $\vec{E} = \mathrm{Re}\left(\vec{E}_0 e^{-i\omega t}\right)$. Making the ansatz $\vec{p}(t) = \mathrm{Re}\left(\vec{p}(\omega)e^{-i\omega t}\right)$ results in an expression for the charge current $\vec{j} = -en\vec{p} = \mathrm{Re}\left(-en\vec{p}(\omega)e^{-i\omega t}\right)$. From the definition of optical conductivity

$$\vec{j}(\omega) = \sigma(\omega)\vec{E}(\omega) \tag{6.48}$$

one can compute the real and imaginary part of optical conductivity

$$\mathrm{Re}\left(\sigma(\omega)\right) = \frac{\sigma_0}{1+\omega^2\tau^2}\,. \tag{6.49}$$

The behaviour for small ω is called the Drude peak, and is presented in figure 6.6. Note the similarity between this figure and 6.4 for small ω. We can thus conclude that in the probe limit, coupling (6.4) introduced a finite-width Drude peak in optical conductivity at the boundary. Drude peak is present in real materials, so the torsion field and the nonminimal coupling in (6.4) create a more realistic model of holographic material in the probe limit.

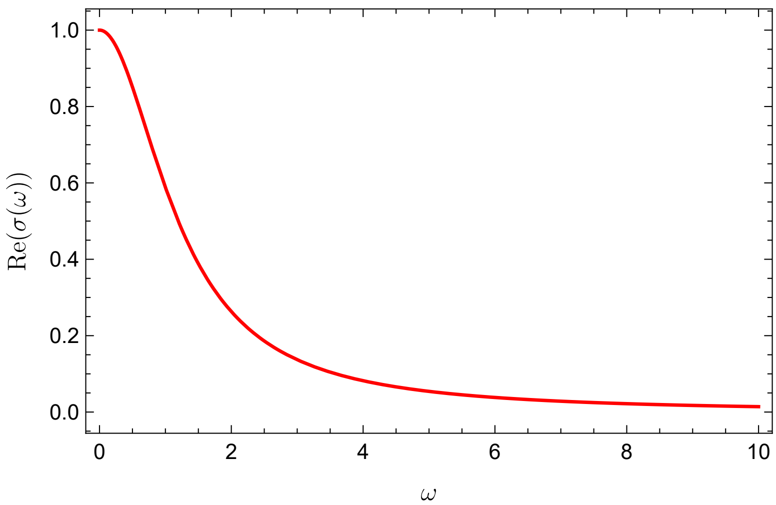

Figure 6.6: Plot of one-sided Lorentzian: the Drude peak, obtained in the Drude model of conductivity.

As already mentioned, linear in ω behaviour for large frequencies is dictated by the underlying conformal invariance in four dimensions. Interestingly, we can find examples of real-world materials that show similar behaviour. For example, the material Ir_2In_8Se was shown in [146] to have a linear in ω behaviour for real part of optical conductivity in a large range of frequencies, while for small ω this material's conductivity profile shows the Drude peak. This material is an example of a semimetal, and one cannot notice a similarity between our results for the $\mathrm{Re}(\sigma(\omega))$ and the one experimentally obtained in [146]. Of course, it is impossible to claim that our simplified model has anything to do with this real-life material, but some general features are obviously present both in our results and in something that we can find in the laboratory, thus accomplishing one of our goals for this thesis: to find the role of Riemann-Cartan geometry outside the standard high-energy phenomenology.

Chapter 7

Noncommutative gravity

This chapter is based on author's work [114, 147, 148, 149].

So far in this thesis, we have assumed that the bulk spacetime is well described by classical geometry, while the novelty of the approach, compared to the more traditional usage of gauge/-gravity duality, was that the bulk is not restricted to be described with Riemannian geometry, with metric as the only fundamental field. However, we would like to make a step ahead and be able to consider quantum bulk spacetimes. We know that holographic duality implies that the bulk spacetime is emergent. It is then natural to assume that, at least in some regime, it makes sense to consider a quantum bulk. As emphasised in [47], one way to introduce a quantum bulk is precisely by using noncommutative geometry, the idea that spacetime coordinates do not commute. We stress that there is no unique approach to defining physical models on noncommutative spacetimes. Most of the work done in this direction studies quantum field theory on deformed Minkowski spacetime. However, for the purpose of holographic duality, we are interested in asymptotically AdS spacetimes and, therefore, should discuss different ways to deform those spaces. In this chapter, we will show two different approaches that enable us to do so, and, as expected, they will be closely related to the first-order formulation of gravity.

7.1 Multiple approaches: Seiberg-Witten map and frame formalism

Even though it is not necessarily connected to string theory, noncommutative physics benefited from string theoretic considerations [150]. Effective physics on a brane, in the presence of a nonzero B field, is described using NC gauge theory. For this reason, Seiberg and Witten considered noncommutative gauge theories and established a mapping between NC degrees of freedom and their commutative counterparts, known as the Seiberg-Witten map. As the goal of this thesis is not to develop NC field theories, but rather to consider their interplay with holographic duality, we will not spend too much time elaborating on the construction and mathematical formalism underlying NC gravity. Instead, we will give a practical overview of the two approaches we use in this work.

As explained in the introductory subsection 1.2.3, in order to deal with NC spacetime and NC field theory, with a canonical choice of commutation relations among coordinates, we can employ the formalism of the Moyal star product (1.30). The simplest choice then is to take a commutative field theory and change each product to the star product. For example, an interacting scalar field theory with self-interaction ϕ^4 is described using the Lagrangian

$$\frac{1}{2}\int \mathrm{d}^D x \, \left(\partial_\mu\phi\star\partial^\mu\phi - m^2\phi\star\phi + \frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\right) . \tag{7.1}$$

One can then expand this action in powers of $\theta^{\mu\nu}$, with the zeroth order manifestly providing us with the commutative theory. Higher orders in the NC parameter give corrections to the commutative theory, and we are precisely interested in them. However, the issue arises when we consider gauge theory. If a commutative theory is invariant under certain gauge transformations (redundancies, in this context), we expect that an analogous notion can be found in the NC theory. However, as we shall see below, there are problems if one uses the simplest logic of substituting ordinary products with the star products as before. It is precisely at this point that one can use the Seiberg-Witten (SW) map. Furthermore, as we already saw in this thesis, certain gravity models can be modeled using gauge theory, so that the combination of the SW map and gauge theory of gravity can be used to define an NC gravity model. This way of reasoning has its own pros and cons. On the one hand, it provides us with a relatively simple model with deformed gravity equations, solving which we can find the corrections to the observed quantities. Furthermore, as the modes manifestly reduce to the commutative one when $\theta \to 0$, corrections are usually treated using standard techniques of field theory. While this comment is generally considered a positive aspect, we stress that it makes certain important questions about the quantum spacetime non-transparent. For example, in this approach, it is rather trivial to define the boundary of a spacetime, because we can use the standard methodology of manifolds [46], but if spacetime coordinates are viewed as operators, this methodology cannot be applied. In this thesis, we shall consider additional formalism that directly defines quantum NC spacetime, with coordinates being operators (acting in a certain representation on Hilbert space). The formalism we will use is called the *frame* formalism, and relies on the first-order formulation of gravity. Comparison between the SW approach and the frame formalism, with many references, can be found in [83]. In this case, we will be interested in reproducing the calculation of the boundary correlation functions of operators dual to the scalar field from the beginning of the section 3.1, modeling the bulk as a quantum AdS spacetime using the frame formalism.

7.2 SW map, twist deformation and CS gravity

The purpose of this section is to introduce the concept of twist deformations in field theory and Seriberg-Witten mapping. We shall not go into the mathematical details of the formalism (usually formulated using Hopf algebras), but rather we will give a practical introduction to the subject, focusing on the main results of the program and applications in gravity. We shall then present some results we obtained using this formalism. We already discussed in the introductory subsection 1.2.3 that one way to incorporate NC effects is to deform the algebra of functions on a classical spacetime. Furthermore, we already encountered one particular deformation of the Minkowski plane: a Moyal-Weyl deformation with constant noncommutativity in Cartesian coordinates (1.30).

The most straightforward way to generalise this is to consider a more general type of twist deformations. Let X_I be a set of mutually commuting vector fields on some manifold $\mathcal{M}$. It is not necessary that their number coincides with the dimension of the spacetime. We then define the twist as

$$\mathcal{F} = \exp\Big(-\tfrac{i}{2}\theta^{IJ} X_I \otimes X_J\Big) . \tag{7.2}$$

The Abelian nature of the vector fields X_I ensures that this twist will be associative. Expanding the exponent of the twist's inverse, we get

$$\mathcal{F}^{-1} = \exp\Big(\tfrac{i}{2}\theta^{IJ} X_I \otimes X_J\Big) = 1\otimes 1 + \tfrac{i}{2}\theta^{IJ} X_I \otimes X_J - \tfrac{1}{8}\theta^{I_1J_1}\theta^{I_2J_2} X_{I_1}X_{I_2} \otimes X_{J_1}X_{J_2} + \cdots\,, \tag{7.3}$$

where μ is the multiplication of the tensor product factors. Acting on salar functions, we have

$$\begin{aligned}(f\star g)(x) &= \mu\left(\mathcal{F}^{-1}(f\otimes g)\right)\\ &= f(x)g(x) + \frac{i}{2}\theta^{IJ} X_I[f]X_J[g] + \mathcal{O}(\theta^2),\end{aligned} \tag{7.4}$$

with $X_I[f] = X_I^\mu \partial_\mu f$. Often, the twist is abbreviated as

$$\mathcal{F}^{-1} \equiv \bar{f}^\alpha \otimes \bar{f}_\alpha\,, \tag{7.5}$$

such that we can deform the commutative algebra of functions on the manifold $\mathcal{M}$ into a more general NC algebra, using the star product

$$f\star g = \bar{f}^\alpha(f)\bar{f}_\alpha(h) = \mu\Big(\mathcal{F}^{-1}(f\otimes g)\Big)\,. \tag{7.6}$$

If we take X_I to be a set of coordinate basis vector fields ∂_μ, we get (1.30). As most of the work in this thesis is carried out using differential forms, we note that the twist can be used to deform the exterior derivative of two forms as

$$\begin{aligned}\tau\wedge_\star\tau' = \bar{f}^\alpha(\tau)\wedge\bar{f}_\alpha(\tau') &= \sum_{n=0}^{+\infty}\left(\frac{i}{2}\right)^n \theta^{I_1J_1}\ldots\theta^{I_nJ_n}(\mathcal{L}_{I_1}\ldots\mathcal{L}_{I_n}\tau_p)\wedge(\mathcal{L}_{J_1}\ldots\mathcal{L}_{J_n}\tau')\\ &= \tau\wedge\tau' + \frac{i}{2}\theta^{IJ}(\mathcal{L}_I\tau)\wedge(\mathcal{L}_J\tau') + \frac{1}{2!}\left(\frac{i}{2}\right)^2\theta^{I_1J_1}\theta^{I_2J_2}(\mathcal{L}_{I_1}\mathcal{L}_{I_2}\tau)\wedge(\mathcal{L}_{J_1}\mathcal{L}_{J_2}\tau') + \ldots\end{aligned} \tag{7.7}$$

In this expression, $\mathcal{L}_I$ stands for the Lie derivative with respect to the vector field X_I.

The next important aspect we shall discuss is the Seiberg-Witten mapping, used to address the issue of NC gauge field theory. Let us consider a classical gauge transformation with parameter $\epsilon = \epsilon^K T_K$, where T_K are algebra generators. We have

$$\delta_\epsilon A = -\mathrm{d}\epsilon - [A,\epsilon] = -\mathrm{d}\epsilon - A\wedge\epsilon + \epsilon\wedge A\,, \tag{7.8}$$

$$\delta_\epsilon F = [\epsilon, F] = \epsilon\wedge F - F\wedge\epsilon\,. \tag{7.9}$$

It is an important property of the commutative gauge transformations that they are closed under the commutator, meaning that we have

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]F = \delta_{-[\epsilon_1,\epsilon_2]}F\,. \tag{7.10}$$

However, if we naively define the star commutator $[\hat{A},\hat{B}]_\star = \hat{A}\star\hat{B} - \hat{B}\star\hat{A}$, and consider NC gauge transformations

$$\widehat{\delta}_{\widehat{\epsilon}}\widehat{A} = -\mathrm{d}\widehat{\epsilon} - \widehat{A}\wedge_\star\widehat{\epsilon} + \widehat{\epsilon}\wedge_\star\widehat{A}\,, \tag{7.11}$$

$$\widehat{\delta}_{\widehat{\epsilon}}\widehat{F} = \widehat{\epsilon}\wedge_\star\widehat{F} - \widehat{F}\wedge_\star\widehat{\epsilon}\,. \tag{7.12}$$

one can easily check that we have

$$[\widehat{\delta}_{\widehat{\epsilon}_1},\widehat{\delta}_{\widehat{\epsilon}_2}]\widehat{F} = \widehat{\delta}_{-[\widehat{\epsilon}_1,\widehat{\epsilon}_2]_\star}\widehat{F} = -[[\widehat{\epsilon}_1,\widehat{\epsilon}_2]_\star,\widehat{F}]_\star\,. \tag{7.13}$$

It is important to note that in the last formula, we have

$$[\widehat{\epsilon}_1, \widehat{\epsilon}_2]_\star = \frac{1}{2}\left([\widehat{\epsilon}_1^K, \widehat{\epsilon}_2^L]_\star \{T_K, T_L\} + \{\widehat{\epsilon}_1^K, \widehat{\epsilon}_2^L\}_\star [T_K, T_L]\right) . \tag{7.14}$$

For a generic Lie algebra (for example, $SU(N)$), the anticommutator of two algebra elements does not belong to the algebra. In order to circumvent this issue, one has to introduce the *universal enveloping algebra* (UEA), which, in addition to the original generators T^A, also contains all the symmetrized powers of them. This would imply that our gauge connection, gauge curvature, and the parameter $\hat{\epsilon}$ are UEA-valued. However, UEA's are infinitely dimensional, and it is not clear what to do with the infinite amount of newly introduced degrees of freedom. It is precisely at this point that the results of [150] become important. It was shown in the quoted paper that one can make a correspondence between an NC theory and an appropriate commutative gauge theory, such that the NC gauge transformations are induced by the commutative ones. In formulas, we have

$$\widehat{\delta}_{\widehat{\epsilon}} \widehat{A}(A) = \widehat{A}(A + \delta_\epsilon A) - \widehat{A}(A) . \tag{7.15}$$

Combining (7.8) and (7.11), we can obtain a perturbative in θ^{IJ} expansion of NC fields, starting from the commutative ones, as

$$\hat{A} = A - \frac{i}{4}\theta^{IJ}\{A_I, \mathcal{L}_J A + F_J\} , \tag{7.16}$$

$$\hat{\epsilon} = \epsilon - \frac{i}{4}\theta^{IJ}\{A_I, \mathcal{L}_J \epsilon\} . \tag{7.17}$$

In this formula we used notation $A_I \equiv i_{X_I}(A) = \frac{1}{2}\Omega_I^{AB} J_{AB} + l^{-1} E_I^A J_{A5}$, representing a contraction with repsect to the vector fields X_I.

Precisely this definition was used in [151] to construct an NC version of the CS action (1.25). The starting point is the observation that in the NC case, we also have

$$\int\limits_{\mathcal{N}_{2n}} \mathrm{Tr}\left(\widehat{F}^{\wedge_\star^n}\right) = \int\limits_{\partial\mathcal{N}_{2n}} \widehat{Q}_{CS}^{(2n-1)} , \tag{7.18}$$

where, in order to introduce the noncommutativity on $\partial\mathcal{N}_{2n}$, it was assumed that the vector fields X_I restrict to vector fields on the boudnary. It turns out that the first-order correction to the commutative CS action is given by

$$\delta_\theta \widehat{Q}_{CS}^{(2n-1)} = \frac{i}{2}\theta^{IJ} \int \mathrm{Tr}\left(F \mathrm{D} F_I \sum_{k=0}^{n-3} (k+1) F^{n-3-k} F_J F^k\right) . \tag{7.19}$$

This form of correction holds for a general Lie group, though we will be interested in the case of the AdS group in order to get corrections to (2.32). Note that for $D = 3$, implying $n = 2$, there are no corrections of any order to the CS action. However, a nontrivial correction is obtained in five dimensions, where we have

$$\delta_\theta L_{CS,NC}^{(5)} = \frac{k}{6}\theta^{IJ} \mathrm{Tr}\left(F \wedge \mathrm{D}F_I \wedge F_J\right) . \tag{7.20}$$

Working with the $SO(4,2)$ Lie algebra, we have

$$\begin{aligned} \mathrm{D}F_I &= \mathrm{d}F_I + [A, F_I] \\ &= \frac{1}{2}\left(\mathrm{D}_\Omega F_I^{AB} + \frac{1}{\ell^2}\left(E^A T_I^B - E^B T_I^A\right)\right) J_{AB} + \frac{1}{\ell}\left(\mathrm{D}_\Omega T_I^A + F_I^{AB} E_B\right) J_{A5} . \end{aligned} \tag{7.21}$$

In order to make the difference between the $SO(4,2)$ covariant derivative and the covariant derivative with respect to the spin-connection, the latter was denoted by D_Ω. However, in what follows, we shall omit the symbol Ω, as we will not use $SO(4,2)$ covariant derivative anymore. Performing a concrete computation in the representation of $SO(4,2)$ defined in Appendix A.1, we have

$$\begin{aligned}S^{(5)}_{CS,\theta} = S^{(5)}_{CS} + \frac{k\theta^{IJ}}{12}\int\Big(&F^{AB}(F_I)_{BC}(\mathrm{D}_\Omega F_J)^C{}_A + \frac{1}{\ell^2}F^{AB}(F_I)_{BC}(T_J)^C E_A \\ &+ \frac{1}{\ell^2}F^{AB}(T_I)_B(\mathrm{D}_\Omega T_J)_A + \frac{2}{\ell^2}F^{AB}(T_I)_B(F_J)_{AC}E^C + \frac{1}{\ell^2}T^A(T_I)^B(\mathrm{D}_\Omega F_J)_{BA} \\ &+ \frac{1}{\ell^2}T^A(\mathrm{D}_\Omega T_I)^B(F_J)_{BA} + \frac{1}{\ell^2}T_A(F_I)^{AB}(F_J)_{BC}E^C + \frac{2}{\ell^4}T_A(T_I)_B(T_J)^{[B}E^{A]}\Big)\,. \end{aligned} \tag{7.22}$$

where $F^{AB} = R^{AB} + \frac{1}{\ell^2}e^Ae^B$. Note the crucial role played by the torsion in this action. If we were to assume a priori that the geometry is Riemannian (which is not consistent due to torsion being part of gauge curvature, but often people make similar assumptions), the first-order correction in (7.22) would vanish, up to boundary terms.

7.2.1 Chiral gravitational anomaly: demonstrating the role of torsion

Similar to considerations in Chapter 4, we can define a four-dimensional gravity model by performing a dimensional reduction of (7.22). The procedure is analogous to the one we used before. However, note that the CTG gravity and its NC extensions were previously considered in the literature (see [147] for a review), because of the relation with EH action through the MMCSW, see subsection 4.1.2. It turns out that the first nontrivial correction to the CTG action is proportional to θ^2, and therefore very cumbersome. In order to get the first-order correction to this action, we shall insist that in the dimensional reduction, one of the vector fields X_I is given by ∂_4, with the additional fifth dimension labeled by the index 4. Interestingly, there indeed is a first-order correction to the CTG action obtained in this way, but it is still very complicated. To make the expressions even simpler, and also to make the connection with more realistic modes, we will perform a symmetry breaking discussed in the subsection 4.1.2. We have to compute all the components appearing in the first-order correction from (7.22). We start by noting that we have

$$\begin{aligned} F^{ab}_{\mu\nu} &= R^{ab}_{\mu\nu} + \frac{1}{\ell^2}(e^a_\mu e^b_\nu - e^a_\nu e^b_\mu)\,, \\ F^{ab}_{\mu 4} &= F^{a4}_{\mu\nu} = 0\,, \\ F^{a4}_{\mu 4} &= -\frac{1}{l^2}\mathrm{D}_\mu\phi^a + \frac{1}{\ell^3}e^a_\mu\varphi \xrightarrow{\text{SB}} \frac{1}{\ell^2}e^a_\mu\,, \end{aligned} \tag{7.23}$$

where indices μ and a stand for four-dimensional coordinate and local Lorentz indices. In the last step, we performed the symmetry breaking (4.37). As for the components of torsion, we have

$$\begin{aligned} T^a_{\mu\nu} &= \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^a_{\mu b}e^b_\nu - \omega^a_{\nu b}e^b_\mu\,, \\ T^a_{\mu 4} &= T^4_{\mu\nu} = 0\,, \\ T^4_{\mu 4} &= \frac{1}{\ell}D_\mu\varphi + \frac{1}{\ell^2}e^a_\mu\phi_a \xrightarrow{\text{SB}} 0\,. \end{aligned} \tag{7.24}$$

Furthermore, we have

$$\begin{aligned}(\mathrm{D}F_I)^{AB} =& \mathrm{d}F_I^{AB} + \Omega^A{}_C F_I^{CB} - \Omega^B{}_C F_I^{CA} \\ =& \frac{1}{2}\Big[\partial_{\widetilde{\mu}}(X_I^{\widetilde{\alpha}} F_{\widetilde{\alpha}\widetilde{\nu}}^{AB}) - \partial_{\widetilde{\nu}}(X_I^{\widetilde{\alpha}} F_{\widetilde{\alpha}\widetilde{\mu}}^{AB}) + X_I^{\widetilde{\alpha}}\Big(\Omega_{\widetilde{\mu}}^A{}_C F_{\widetilde{\alpha}\widetilde{\nu}}^{CB} - \Omega_{\widetilde{\mu}}^B{}_C F_{\widetilde{\alpha}\widetilde{\nu}}^{CA} \\ & - \Omega_{\widetilde{\nu}}^A{}_C F_{\widetilde{\alpha}\widetilde{\mu}}^{CB} + \Omega_{\widetilde{\nu}}^B{}_C F_{\widetilde{\alpha}\widetilde{\mu}}^{CA}\Big)\Big]\mathrm{d}x^{\widetilde{\mu}}\mathrm{d}x^{\widetilde{\nu}}\,. \end{aligned} \tag{7.25}$$

Writing the components explicitly, we have

$$\begin{aligned}(\mathrm{D}F_I)_{\mu\nu}^{ab} &= \mathrm{D}_\mu(X_I^\alpha F_{\alpha\nu}^{ab}) - \mathrm{D}_\nu(X_I^\alpha F_{\alpha\mu}^{ab})\,, \\ (\mathrm{D}F_I)_{\mu\nu}^{a4} &= -\mathrm{D}_\mu(X_I^4 F_{\nu 4}^{a4}) + \mathrm{D}_\nu(X_I^4 F_{\mu 4}^{a4}) \xrightarrow{\mathrm{SB}} -\frac{1}{\ell^2}\left(\mathrm{D}_\mu(X_I^4 e_\nu^a) - \mathrm{D}_\nu(X_I^4 e_\mu^a)\right)\,, \\ (\mathrm{D}F_I)_{\mu 4}^{ab} &= \frac{1}{\ell^2}X_I^4\left(\phi^a F_{\mu 4}^{b4} - \phi^b F_{\mu 4}^{a4}\right) \xrightarrow{\mathrm{SB}} 0\,, \\ (\mathrm{D}_\Omega F_I)_{\mu 4}^{a4} &= \mathrm{D}_\mu(X_I^\alpha F_{\alpha 4}^{a4}) - \frac{1}{l^2}X_I^\alpha F_{\alpha\mu}^{ab}\phi_b \xrightarrow{\mathrm{SB}} \frac{1}{\ell^2}\mathrm{D}_\mu(X_I^\alpha e_\alpha^a)\,. \end{aligned} \tag{7.26}$$

In addition, we compute the derivatives of contracted torsion as

$$\begin{aligned}(\mathrm{D}T_I)^A &= \mathrm{d}T_I^A + \Omega^A{}_B T_I^B \\ &= \frac{1}{2}\left[\partial_{\widetilde{\mu}}\left(X_I^{\widetilde{\alpha}} T_{\widetilde{\alpha}\widetilde{\nu}}^A\right) - \partial_{\widetilde{\nu}}\left(X_I^{\widetilde{\alpha}} T_{\widetilde{\alpha}\widetilde{\mu}}^A\right) + X_I^{\widetilde{\alpha}}\left(\Omega_{\widetilde{\mu}}^A{}_B T_{\widetilde{\alpha}\widetilde{\nu}}^B - \Omega_{\widetilde{\nu}}^A{}_B T_{\widetilde{\alpha}\widetilde{\mu}}^B\right)\right]\mathrm{d}x^{\widetilde{\mu}}\mathrm{d}x^{\widetilde{\nu}}\,, \end{aligned} \tag{7.27}$$

which in components reads

$$(\mathrm{D}T_I)_{\mu\nu}^a = \mathrm{D}_\mu(X_I^\alpha T_{\alpha\nu}^a) - \mathrm{D}_\nu(X_I^\alpha T_{\alpha\mu}^a)\,, \tag{7.28}$$

$$(\mathrm{D}T_I)_{\mu\nu}^4 = \mathrm{D}_\mu(X_I^4 T_{4\nu}^4) - \mathrm{D}_\nu(X_I^4 T_{4\mu}^4) \xrightarrow{\mathrm{SB}} 0\,, \tag{7.29}$$

$$(\mathrm{D}T_I)_{\mu 4}^a = -X_I^4 T_{\mu 4}^4 \phi^a \xrightarrow{\mathrm{SB}} 0\,, \tag{7.30}$$

$$(\mathrm{D}T_I)_{\mu 4}^4 = \mathrm{D}_\mu(X_I^\alpha T_{\alpha 4}^4) - X_I^\alpha T_{\alpha\mu}^a\phi_a \xrightarrow{\mathrm{SB}} 0\,. \tag{7.31}$$

Combining all previous formulas, we obtain the form of the reduced action, which, after the symmetry breaking, takes the form of

$$\begin{aligned}S_{red,\theta} = \frac{(2\pi R)k}{12}\theta^{IJ}\int\Bigg[&\frac{2}{\ell^4}X_J^4 R^{ab}T_a(e_I)_b - \frac{4}{\ell^4}X_J^4 T^a(R_I)_{ab}e^b + \frac{2}{\ell^4}X_J^4 R^{ab}(T_I)_a e_b \\ &+ \frac{6}{\ell^6}X_J^4 T^a e_a(e_I)^b e_b - \frac{4}{\ell^4}(\partial X_J^4)R^{ab}e_b(e_I)_a - \frac{2}{\ell^4}(\partial X_J^4)(R^{ab}e_b)_I e_a\Bigg]\,. \end{aligned} \tag{7.32}$$

Let us explicitly illustrate the reduction procedure for one of the terms in (7.22): the fourth one, for example. We have

$$\begin{aligned}&-\frac{\theta^{IJ}}{\ell^2}X_I^{\widetilde{\alpha}}X_J^{\widetilde{\beta}}F_{\widetilde{\mu}\widetilde{\nu}}^{AB}T_{\widetilde{\alpha}\widetilde{\rho}B}F_{\widetilde{\beta}\widetilde{\sigma}AC}E_{\widetilde{\tau}}^C\varepsilon^{\widetilde{\mu}\widetilde{\nu}\widetilde{\rho}\widetilde{\sigma}\widetilde{\tau}}\mathrm{d}^4x \\ &= -\frac{\theta^{IJ}}{\ell^2}\Big(X_I^\alpha X_J^\beta F_{\widetilde{\mu}\widetilde{\nu}}^{AB}T_{\alpha\widetilde{\rho}B}F_{\beta\widetilde{\sigma}AC}E_{\widetilde{\tau}}^C + X_I^4 X_J^4 F_{\widetilde{\mu}\widetilde{\nu}}^{AB}T_{4\widetilde{\rho}B}F_{4\widetilde{\sigma}AC}E_{\widetilde{\tau}}^C + X_I^\alpha X_J^4 F_{\widetilde{\mu}\widetilde{\nu}}^{AB}T_{\alpha\widetilde{\rho}B}F_{4\widetilde{\sigma}AC}E_{\widetilde{\tau}}^C \\ &\quad + X_I^4 X_J^\beta F_{\widetilde{\mu}\widetilde{\nu}}^{AB}T_{4\widetilde{\rho}B}F_{\beta\widetilde{\sigma}AC}E_{\widetilde{\tau}}^C\Big)\varepsilon^{\widetilde{\mu}\widetilde{\nu}\widetilde{\rho}\widetilde{\sigma}\widetilde{\tau}}\mathrm{d}^4x\,. \end{aligned} \tag{7.33}$$

Unpacking even further, we have

$$\begin{aligned}
&-\frac{\theta^{IJ}}{\ell^2}X_I^\alpha X_J^4\Big(F^{AB}_{\widetilde{\mu}\widetilde{\nu}}T_{\alpha\widetilde{\rho}B}F_{4\widetilde{\sigma}AC}E^C_{\widetilde{\tau}}-F^{AB}_{\widetilde{\mu}\widetilde{\nu}}T_{4\widetilde{\rho}B}F_{\alpha\widetilde{\sigma}AC}E^C_{\widetilde{\tau}}\Big)\varepsilon^{\widetilde{\mu}\widetilde{\nu}\widetilde{\rho}\widetilde{\sigma}\widetilde{\tau}}\mathrm{d}^4x\\
&=\frac{\theta^{IJ}}{\ell^2}X_I^\alpha X_J^4\Big(-F^{AB}_{\mu\nu}T_{\alpha\rho B}F_{4\sigma AC}E^C_4+2F^{AB}_{\mu4}T_{\alpha\nu B}F_{4\rho AC}E^C_\sigma-2F^{AB}_{\mu4}T_{4\nu B}F_{\alpha\rho AC}E^C_\sigma\Big)\varepsilon^{\mu\nu\rho\sigma}\mathrm{d}^4x\\
&=\frac{\theta^{IJ}}{\ell^2}X_I^\alpha X_J^4\Big(F^{ab}_{\mu\nu}T_{\alpha\rho b}F_{\sigma4a4}E^4_4-2F^{b4}_{\mu4}T_{\alpha\nu b}F_{\rho4c4}E^c_\sigma+2F^{a4}_{\mu4}T^4_{\nu4}F_{\alpha\rho ac}E^c_\sigma\Big)\varepsilon^{\mu\nu\rho\sigma}\mathrm{d}^4x\\
&\xrightarrow{\text{SB}}-\frac{\theta^{IJ}}{\ell^4}X_I^\alpha X_J^4R^{ab}_{\mu\nu}T_{\alpha\rho a}e_{\sigma b}\varepsilon^{\mu\nu\rho\sigma}\mathrm{d}^4x\,.
\end{aligned}\tag{7.34}$$

where the last line follows after the symmetry breaking, obtaining the promised fourth term in (7.22). Assuming that $\partial_\mu X_i^4=0$, which does not break four-dimensional diffeomorphisms, we obtain the final form of the action

$$\begin{aligned}
S_{red,NC}=&\frac{(2\pi R)k}{8\ell^3}\int\varepsilon_{abcd}\left[\ell^2R^{ab}R^{cd}+2R^{ab}e^ce^d+\frac{1}{\ell^2}e^ae^be^ce^d\right]\\
&+\frac{(2\pi R)k}{12}\theta^{I4}\int\left[\frac{2}{\ell^4}R^{ab}T_a(e_I)_b-\frac{4}{\ell^4}T^a(R_I)_{ab}e^b+\frac{2}{\ell^4}R^{ab}(T_I)_ae_b+\frac{6}{\ell^6}T^ae_a(e_I)^be_b\right].
\end{aligned}\tag{7.35}$$

Again, note that by setting $T^a=0$ in the action (7.35), the correction vanishes, indicating the close relation between NC corrections in this model and torsion. The equations of motion of (7.35), obtained by varying with respect to the vielbein and the spin-connection, yield

$$\begin{aligned}
\varepsilon_{abc}{}^d\left(R^{ab}e^c+\frac{1}{\ell^2}e^ae^be^c\right)-\frac{\theta^{I4}}{3\ell}\left[\left(R^{db}+\frac{3}{\ell^2}e^de^b\right)(\mathrm{D}e_I)_b-2(\mathrm{D}R_I)^{db}e_b\right]&=0\,,\\
\varepsilon^{ac}{}_{bd}T^be^d+\frac{\theta^{I4}}{3\ell}\left[\frac{1}{2}R^{ab}e^c(e_I)_b-\frac{1}{2}R^{cb}e^a(e_I)_b+(R_I)^{ab}e^ce_b-(R_I)^{cb}e^ae_b+\frac{3}{\ell^2}e^ae^be^c(e_I)_b\right]&=0\,.
\end{aligned}\tag{7.36}$$

We simplified the equations as we know that the classical solution to the EH equations without any matter context is torsionless, and therefore we can neglect all the terms containing the torsion field in the first-order corrections from (7.36). Working perturbatively in powers of θ^{I4}, one can show that a pure AdS solution without torsion remains the solution of (7.36). In addition, we can consider a BH solution. Up to this point, we didn't pay much attention to the AdS Schwarzschild black hole (2.12). This spacetime solves the commutative equations of motion for the EH theory. We therefore seek a modification of the commutative geometry by writing $e^a\to e^a+\delta e^a$ and $\omega^{ab}\to\omega^{ab}+\delta\omega^{ab}$. Next, we note that so far we have not specified vector fields X_I. In order to get a tractable calculation, we choose that the only nonvanishing vector field (apart from already introduced ∂_4) is given by $X_r=\partial_r$, with r being the black-hole radial variable. One can then verify that the equations are solved if one takes $\delta e^a=0$ and

$$\delta\omega_0^{23}=\frac{m\theta^{14}}{\ell r^3}\,,\qquad\delta\omega_2^{03}=-\frac{m\theta^{14}}{2\ell r^2f(r)}\,,\qquad\delta\omega_3^{02}=\frac{m\theta^{14}}{2\ell r^2f(r)}\sin\theta\,,\tag{7.37}$$

where $f^2(r)\equiv\left(1-\frac{2m}{r}+\frac{r^2}{\ell^2}\right)$. For this solution, torsion is given as

$$\tilde{T}^0_{23}=-\frac{m\theta^{14}}{\ell}\frac{\sin\theta}{rf(r)}\,,\qquad\tilde{T}^2_{03}=\frac{m\theta^{14}}{2\ell}\frac{\sin\theta}{r^2}\,,\qquad\tilde{T}^3_{02}=-\frac{m\theta^{14}}{2\ell r^2}\,.\tag{7.38}$$

This implies that the underlying geometry is indeed the Riemann-Cartan geometry. Components of the Riemann-Cartan curvature are

$$\tilde{R}^{01}_{23} = -\frac{m\theta^{14}}{\ell}\frac{\sin\theta}{r^2}\,, \qquad \tilde{R}^{02}_{13} = -\frac{m\theta^{14}}{\ell}\left[\frac{f'(r)}{2r^2} + \frac{f(r)}{r^3}\right]\frac{\sin\theta}{f^2(r)} = -\sin\theta\tilde{R}^{03}_{12}\,,$$
$$\tilde{R}^{12}_{03} = \frac{m\theta^{14}}{\ell}\left[\frac{f'(r)}{2r^2} - \frac{f(r)}{r^3}\right]\sin\theta = -\sin\theta\tilde{R}^{13}_{02}\,, \qquad \tilde{R}^{23}_{01} = \frac{3m\theta^{14}}{\ell r^4}\,. \tag{7.39}$$

One particular aspect of this spacetime configuration is that the Pontryagin density $\mathcal{P} = R^{ab}R_{ab}$ is nonvanishing. Making the explicit calculation, we obtain

$$\begin{aligned} R^{ab}R_{ab} &= 2\big(-R^{01}R^{01} - R^{02}R^{02} - R^{03}R^{03} + R^{12}R^{12} + R^{13}R^{13}R^{23}R^{23}\big) \\ &= \Big(\big(\frac{8m}{r^3} - \frac{4}{\ell^2}\big)\frac{4m\theta^{14}}{\ell r^2}\sin\theta + 8\big(\frac{m}{r^3} + \frac{1}{\ell^2}\big)\frac{m\theta^{14}r\sin\theta}{f(r)\ell}\big(\frac{f'(r)}{2r^2} + \frac{f(r)}{r^3} - \frac{f'(r)}{2r^2} + \frac{f(r)}{r^3}\big)\Big)\mathrm{d}^4x \\ &= \frac{48m^2\theta^{14}}{\ell r^5}\sin\theta\,\mathrm{d}^4x\,. \end{aligned} \tag{7.40}$$

This is interesting because $\mathcal{P}$ appears in the expression for axial gravitational anomaly. Namely, as we already discussed in Chapter 6, the Dirac action has two conserved currents J^μ and $\mathcal{J}^\mu_{\text{ax}}$. In the case of an applied electromagnetic field, both of them cannot be simultaneously conserved at the quantum level. However, a similar situation happens if the Dirac action is considered on a curved background [43]. It turns out that the non-conservation of J^5 current is given as

$$\mathrm{d}\star\mathcal{J}_{\text{ax}} = \frac{1}{96\pi^2}R^{ab}R_{ab}\,, \tag{7.41}$$

where this result is standardly derived in the realm of Riemannian geometry. In the case of Riemann-Cartan geometry, the generalization of (7.41) was done for the first time in [152], where it was shown that the Nieh-Yan term can modify the expression for the chiral anomaly. Interestingly, the original derivation was challenged in subsequent years [153], with many proposals for the correct expression of the anomaly (see [154] for a recent discussion). Luckily, in our case, the Nieh-Yan term $\mathcal{N} = T^aT_a - R^{ab}e_ae_b = 0$, so we do not have to worry about this issue. Therefore, in the case at hand, we have

$$\mathrm{d}\star\mathcal{J}_{\text{ax}} = \frac{m^2\theta^{14}}{2\pi^2\ell r^5}\sin\theta\;\mathrm{d}t\wedge\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi\,, \tag{7.42}$$

or, equivalently

$$\partial_\mu(\sqrt{-g}\mathcal{J}^\mu_{\text{ax}}) = \frac{m^2\theta^{14}}{2\pi^2\ell r^5}\sin\theta\,. \tag{7.43}$$

We see that the NC effects, parametrized by θ^{14}, enable us to obtain a BH spacetime for which the left-hand side of (7.43) is nonvanishing

7.3 Scalar field with twist deformation on asymptotically AdS spacetime

Let us now discuss another way to use NC theory to obtain some interesting conclusions about quantum deformations of spacetime. In the last section, we have seen that it is possible to use the SW map to obtain the corrections to classical geometry. However, those corrections depend upon our choice of the twist vector fields X_I, and it is hard to find any consistent choice of the vector fields in a general situation. This is not to say that there are not certain situations where a preferred choice exists [155]. It was argued in this paper that the choice of the twist vector fields (coordinate system corresponding to the coordinate basis in which we have the canonical commutation relations) should be associated with Fermi normal coordinates, defined by a freely falling observer. Unfortunately, any computation using those vector fields is very complicated, and we therefore analyse a different situation where we can analytically calculate some modifications of an asymptotically AdS spacetime. We will use the methodology of [156], where the idea is that, in certain situations, the equations of motion for the NC scalar (or Dirac) field can be written as a classical (commutative) equation in a modified, *effective* metric. Historically, this is not the first paper to promote such ideas; see also [157], but our goal is precisely to extend the results of [156] to the realm of asymptotically AdS spacetime. For this, let us start from the five-dimensional Reissner–Nordström AdS black hole, whose line element is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{M}{r^2} + \frac{Q^2}{r^4} + \frac{r^2}{\ell^2}\right)\mathrm{d}t^2 + \frac{1}{\left(1 - \frac{M}{r^2} + \frac{Q^2}{r^4} + \frac{r^2}{\ell^2}\right)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2 \,. \tag{7.44}$$

This spacetime solves Einstein's equation with a negative cosmological constant and a gauge field of the form

$$A = -\frac{\sqrt{3}Q}{2r^2}\,\mathrm{d}t \,. \tag{7.45}$$

We consider a massless charged scalar field f in this geometry, with classical action of the form

$$S = \int \mathrm{d}x^D \sqrt{-g}\, g^{\mu\nu} D_\mu f^\dagger D_\nu f \,, \tag{7.46}$$

where $D_\mu = \partial_\mu - iqA_\mu$. In order to define the NC action, we use the twist formalism already introduced in the last section. There is one important difference. In the previous section, we deformed the gravity action and obtained corrections to Einstein's equations. We used the first-order formulation of gravity not only because of our interest in gauge theories of gravity, but also because, as hinted at in the introductory chapter, it is much easier to address NC deformations in the first-order formalism. The reason for this is that second-order gravity action contains the factor $\sqrt{-g}$, and by using the NC field $\hat{g}_{\mu\nu}$, it would be hard to define the square root perturbatively. The same issue is present in (7.46). Luckily, there is a way around this. So far, we assumed that the vector fields X_I are mutually commuting, but no further restriction has been made. However, as action (7.46) is defined on a fixed geometry, we can use Killing vector fields (if any) for the twist vectors. This is because we have

$$g_{\mu\nu} \star (\cdots) = g_{\mu\nu} \cdot (\cdots) + \frac{i}{2}\theta^{IJ} X_I[g_{\mu\nu}]\mathcal{L}_{X_J}(\cdots) \,, \tag{7.47}$$

and if we choose coordinates such that the vector fields X_I are a subset of coordinate vector fields (in which case the metric components do not depend on the respective coordinate), the metric field has a trivial product with any other field. Therefore, we can define NC action as

$$\int \mathrm{d}^4x \sqrt{|g|} \star D_\mu \hat{f}^* \star D_\nu \hat{f} \,, \tag{7.48}$$

where we introduced NC scalar field $\hat{f}$, removed all the star products between the metric field and other fields, and $D_\mu \hat{f} = \partial_\mu \hat{f} - iq\hat{A}_\mu \star \hat{f}$. Next, we make the choice of the twist vector fields.

We choose only two vector fields, $X_1 = \partial_t$ and $X_2 = \partial_\varphi$, where φ is the angular coordinate from $\mathrm{d}\Omega_3^2 = \mathrm{d}\psi^2 + \sin^2\psi(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)$. A concrete form of the twist element we take is

$$\mathcal{F} = e^{-i\frac{\hbar}{2}(\partial_t \otimes \partial_\varphi - \partial_\varphi \otimes \partial_t)}\,, \tag{7.49}$$

where we defined a dimensionful constant $\hbar$ measuring the noncommutativity. Note that, as we are using coordinate vector fields to define the twist, $X_I = \delta_I^\mu \partial_\mu$, we can write a simpler (compared to (7.16)) form of the SW map in the coordinate basis. The NC gauge field is expanded as

$$\hat{A}_\mu = A_\mu - \frac{1}{2}\theta^{\rho\sigma}A_\rho\left(\partial_\sigma A_\mu + F_{\sigma\mu}\right) + \mathcal{O}(\theta^2)\,, \tag{7.50}$$

compare with (7.16). In addition, we need the SW map for the charged scalar field, which is given by

$$\hat{f} = f - \frac{1}{4}\theta^{\mu\nu}A_\mu\left(\partial_\nu f + D_\nu f\right) + \mathcal{O}(\theta^2)\,. \tag{7.51}$$

The action for the NC scalar field is expanded to the first order in $\hbar$ as

$$\int \mathrm{d}^4x\sqrt{|g|}\Big[g^{\mu\nu}D_\mu f^* D_\nu f + \frac{1}{2}\theta^{\alpha\beta}g^{\mu\nu}\Big(-\frac{1}{2}(D_\mu f)^* F_{\alpha\beta}D_\nu f + (D_\mu f)^* F_{\alpha\nu}D_\beta f + (D_\beta f)^* F_{\alpha\mu}D_\nu f\Big)\Big]\,. \tag{7.52}$$

7.3.1 Effective metric

Let us now discuss the consequences of the action (7.52). The equation for the NC scalar field, following from the action (7.52), takes the form of

$$\Big[-\frac{1}{f(r)}\partial_t^2 - \frac{1}{r^2}\frac{i\sqrt{3}Qq}{f(r)}\partial_t + \big(\frac{3}{r} + \frac{5r}{\ell^2} - \frac{M}{r^3} - \frac{Q^2}{r^5}\big)\partial_r + \frac{1}{r^2}\Delta_{S^3} + f(r)\partial_r^2 + \frac{3Q^2q^2}{4r^4 f(r)}\Big]f + \frac{4\sqrt{3}\hbar qQ}{r^4}\Big[\big(\frac{M}{r^2} - \frac{2Q^2}{r^4} + \frac{r^2}{\ell^2}\big)\partial_\varphi + rf(r)\partial_r\partial_\varphi\Big]f = 0\,. \tag{7.53}$$

The first line in (7.53) corresponds to the classical KG equation in curved spacetime and external electric field, while the second line gives the first-order NC corrections. Interestingly, it turns out that, as long as we are interested in the first-order in $\hbar$ corrections, the same equation can be obtained by using the classical equation (the first line of (7.53)), but with an effective metric

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} \frac{M}{r^2} - \frac{Q^2}{r^4} - \frac{r^2}{\ell^2} - 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{-\frac{M}{r^2} + \frac{Q^2}{r^4} + \frac{r^2}{\ell^2} + 1} & 0 & 0 & -\frac{\sqrt{3}\hbar qQ\sin^2(\psi)\sin^2(\theta)}{4r} \\ 0 & 0 & r^2 & 0 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\psi) & 0 \\ 0 & -\frac{\sqrt{3}\hbar Q\sin^2(\psi)\sin^2(\theta)}{4r} & 0 & 0 & r^2\sin^2(\theta)\sin^2(\psi) \end{pmatrix}. \tag{7.54}$$

The modification is present in the off-diagonal components. Unfortunately, we can give no natural interpretation of these off-diagonal components. However, note that in a related work of ours [137], we showed that in the case of Melvin's electric universe, with λ-Minkowski twist, the interpretation of the off-diagonal components appearing in the effective metric is the transition from the static frame to a nonintertial rotating reference frame. In addition, we will show in the next section that in the case of a magnetically charged AdS black hole, nontrivial conclusions about the effective metric can be made.

7.3.2 Planar magnetic RN black hole

In this subsection, we give an example including a four-dimensional AdS RN black hole with magnetic charge Q_m. The line element of this spacetime is given by [158]

$$\mathrm{d}s^2 = -\left(\frac{r^2}{\ell^2} - \frac{M}{r} + \frac{Q_m^2}{r^2}\right)\mathrm{d}t^2 + \frac{1}{\left(\frac{r^2}{\ell^2} - \frac{M}{r} + \frac{Q_m^2}{r^2}\right)}\mathrm{d}r^2 + r^2\left(\mathrm{d}x^2 + \mathrm{d}y^2\right). \tag{7.55}$$

The gauge field, in the Landau gauge, is

$$A = Q_m x\mathrm{d}y \qquad \Rightarrow \qquad F = Q_m \mathrm{d}x \wedge \mathrm{d}y. \tag{7.56}$$

Holographically, this creates a constant magnetic field in the dual theory.The commutative equation for a massless scalar field $f(t,r,x,y)$ is given by

$$\Big(-\frac{r^4}{Q_m^2 - Mr + \frac{r^4}{\ell^2}}\partial_t^2 + \left(Q_m^2 - Mr + \frac{r^4}{\ell^2}\right)\partial_r^2 - 2iQ_m qx\partial_y - \left(M - 4\frac{r^3}{\ell^2}\right)\partial_r + \partial_x^2 + \partial_y^2 - Q_m^2 q^2 x^2\Big) f(t,r,x,y) = 0\,. \tag{7.57}$$

By introducing NC effects as before, now taking $X_1 = \partial_x$ and $X_2 = \partial_y$, the equation of motion, in the first order in $\bar{k}$, takes the following form

$$\begin{aligned}\Big(&-\frac{r^4}{Q_m^2 - Mr + \lambda^2 r^4}\partial_t^2 + \left(Q_m^2 - Mr + \lambda^2 r^4\right)\partial_r^2 - 2iQ_m qx\partial_y - \left(M - 4\lambda^2 r^3\right)\partial_r + \partial_x^2 + \partial_y^2 \\ &- Q_m^2 q^2 x^2 + 2Q_m q\bar{k}\Big(-\frac{r^4\partial_t^2}{Q_m^2 - Mr + \lambda^2 r^4} + \left(Q_m^2 - Mr + \lambda^2 r^4\right)\partial_r^2 + Q_m qx\left(Q_m qx + 2i\partial_y\right) \\ &- \left(M - 4\lambda^2 r^3\right)\partial_r - \partial_x^2 - \partial_y^2\Big)\Big) f(t,r,x,y) = 0\,,\end{aligned} \tag{7.58}$$

where, for brevity, we introduced $\lambda = \frac{1}{\ell}$. This equation can be obtained from an effective metric of the form

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} -\left(\frac{Q_m^2}{r^2} - \frac{M}{r} + \frac{r^2}{\ell^2}\right)(1 - 2\bar{k}Q_m q) & 0 & 0 & 0 \\ 0 & \frac{1-2\bar{k}Q_m q}{\left(\frac{Q_m^2}{r^2} - \frac{M}{r} + \frac{r^2}{\ell^2}\right)} & 0 & 0 \\ 0 & 0 & r^2(2\bar{k}Q_m q + 1) & 0 \\ 0 & 0 & 0 & r^2(2\bar{k}Q_m 1 + 1) \end{pmatrix}. \tag{7.59}$$

Contrary to (7.54), the corrections here enter the diagonal metric components. Changing $x \to \tilde{x} = \sqrt{(2\bar{k}Q_m q + 1)}x$ and $y \to \tilde{y} = \sqrt{(2\bar{k}Q_m q + 1)}y$ creates a transition

$$F = Q_m \mathrm{d}x \wedge \mathrm{d}y = Q_m \frac{1}{(2\bar{k}Q_m q + 1)}\mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y} \approx (Q_m - 2\bar{k}qQ_m^2)\mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y}\,, \tag{7.60}$$

which effectively changes the magnetic flux in the dual field theory. The horizon is still at the same position as in the commutative case. In order to check the temperature in the dual field theory, we denote $\tilde{f}^{-2}(r) = \frac{1-2\bar{k}Q_m q}{\left(\frac{Q_m^2}{r^2} - \frac{M}{r} + \frac{r^2}{\ell^2}\right)}$, so that the metric takes the form of

$$\mathrm{d}s^2 = -\tilde{f}^2(r)(1 - 4\bar{k}Q_m q)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\tilde{f}^2(r)} + r^2(\mathrm{d}\tilde{x}^2 + \tilde{y}^2)\,, \tag{7.61}$$

Then, the same procedure as in Chapter 1 gives that the boundary temperature is the same as in the commutative case $\tilde{T} = T_{\text{com}}$. Alternatively, we may shift the radial variable $r \to \tilde{r} = r(1 + \bar{k}Q_m q)$. In the end, the NC effects are present, so this modification has physical implications. Another, more systematic way to analyse this theory is to consider equation (7.58) and note that the $x - y$ part of this equation can be written as the famous Landau problem of a particle in a constant magnetic field. This type of consideration can play a role in exploring the effect of the magnetic field on a condensate in a holographic superfluid/superconductor. This was recently explored in [159].

7.4 Frame formalism

We saw in previous chapters that the first-order gravity theories can be studied using holography techniques. In this chapter, we have presented results indicating that the first-order formulation of gravity is very useful in defining the NC theory of gravity. It is therefore natural to try to define the boundary dual to the NC gravity theory obtained using the SW map. However, it is much harder to perform the holographic analysis of those gravity theories than one might suspect. First, there is an issue of twist vector fields, and we should impose that they are tangent to the boundary, which is hard to justify a priori. Furthermore, when performing the standard SW expansion, one usually ignores all the boundary terms, which could be relevant for the holographic analysis. We therefore restrict ourselves to a simpler situation to study QFT in AdS spacetime and compute boundary observables. In the last section, we saw how the twist approach can be used to gain some intuition about the boundary dual; yet, no crucial difference between the commutative and noncommutative theory was found. Furthermore, we did not use the first-order formulation for that discussion. In this section, we shall rely on the frame formalism, a particular approach to NC spacetimes [160]. This formalism relies on the first-order formulation, where the role of the commutative frame fields is taken by the momenta $\hat{p}_a$. The quantization principle is to start from the classical relation (2.5) containing differentiation and take

$$e^{\mu}_a(x) = \mathsf{e}_a x^{\mu} \;\mapsto\; e^{\mu}_a(\hat{x}) = [\hat{p}_a, \hat{x}^{\mu}] \, . \tag{7.62}$$

We will take the momenta to form a Lie algebra that precisely corresponds to the isometry algebra of the classical spacetime. It turns out that this way, we will indeed be working with a spacetime with a constant scalar curvature, as AdS spacetime is [160].

7.4.1 Fuzzy AdS_2 and AdS_3

Let us start by describing the Fuzzy AdS_2. This NC spacetime is introduced analogously to the dS_2 from [161]. We are interested in using $\mathfrak{so}(2,1) \equiv \mathfrak{sl}(2,\mathbb{R})$ algebra to reproduce frame relations from (2.6). Using commutation relations (2.27), we see that the choice

$$\hat{z} = i\bar{k}E_+ \, , \qquad \hat{p}_z = H \, , \tag{7.63}$$

$$\hat{t} = -i\bar{k}H \, , \qquad \hat{p}_t = E_+ \, , \tag{7.64}$$

results precisely in

$$[\hat{p}_z, \hat{z}] = \hat{z} \, , \qquad [\hat{p}_t, \hat{t}] = \hat{z} \, . \tag{7.65}$$

We introduced a dimensionful parameter $\bar{k}$ (we use the same symbol as in the last section) to measure the noncommutativity of coordinates, as it is easy to check that from (7.63) follows

$$[\hat{z}, \hat{t}] = i\bar{k}\hat{z} \, . \tag{7.66}$$

In the case of the fuzzy AdS_3, the symmetry algebra is $\mathfrak{so}(2,2) \equiv \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, and therefore we have two copies of the $\mathfrak{sl}(2,\mathbb{R})$ alegbra. If we denote one set of generators as before, and generators from the other $\mathfrak{sl}(2,\mathbb{R})$ algebra with a hat, we can follow the reasoning from [83] to define the fuzzy AdS_3 as

$$\hat{z} = -2i\lambda\!\!\!^{-}\sqrt{E_+\bar{E}_+}\,, \qquad \hat{p}_z = H + \bar{H}\,, \tag{7.67}$$

$$\hat{x} = i\lambda\!\!\!^{-}\left(\sqrt{\frac{\bar{E}_+}{E_+}}\left(H - \frac{1}{4}\right) + \sqrt{\frac{E_+}{\bar{E}_+}}\left(\bar{H} - \frac{1}{4}\right)\right), \qquad \hat{p}_x = E_+ + \bar{E}_+\,, \tag{7.68}$$

$$\hat{t} = i\lambda\!\!\!^{-}\left(\sqrt{\frac{\bar{E}_+}{E_+}}\left(H - \frac{1}{4}\right) - \sqrt{\frac{E_+}{\bar{E}_+}}\left(\bar{H} - \frac{1}{4}\right)\right), \qquad \hat{p}_t = E_+ - \bar{E}_+\,. \tag{7.69}$$

We will be working in a concrete unitary irreducible representation of the $\mathfrak{sl}(2,\mathbb{R})$ algebra, such that the operator E_+ is positive, and the square root is well-defined. Here, unlike in $D = 2$ case, coordinates are not expressed as elements from the algebra, but rather as more complicated functions of generators. One can check that the commutator between two coordinates is not expressable as a function of coordinates solely. However, the parameter $\lambda\!\!\!^{-}$ still controls the noncommutativity of the coordinates, as contraction $\lambda\!\!\!^{-} \to 0$ leads to trivial comutation relations. On the other hand, momenta close in algebra, as expected,

$$[\hat{p}_z, \hat{p}_x] = \hat{p}_x\,, \quad [\hat{p}_z, \hat{p}_t] = \hat{p}_t\,, \quad [\hat{p}_x, \hat{p}_t] = 0\,. \tag{7.70}$$

7.4.2 Scalar field on fuzzy AdS

In this subsection, we consider a classical scalar field on fuzzy AdS spacetime. By this, we assume the solutions of the fuzzy KG equations, defined using the fuzzy Laplacian. For fuzzy AdS_2, the fuzzy Laplacian takes the form

$$\Delta\phi = [\hat{p}_z, [\hat{p}_z, \phi]] - [\hat{p}_z, \phi] - [\hat{p}_t, [\hat{p}_t, \phi]]\,, \tag{7.71}$$

while in three dimensions we have

$$\Delta\phi = [\hat{p}_z, [\hat{p}_z, \phi]] - 2[\hat{p}_z, \phi] - [\hat{p}_t, [\hat{p}_t, \phi]] + [\hat{p}_x, [\hat{p}_x, \phi]]\,. \tag{7.72}$$

NC KG equation then takes the form

$$\Delta\phi = M^2\phi\,. \tag{7.73}$$

Some solutions of these equations can be found by exploring the commutation relations between coordinates and momenta, and making the ansatz about the ordering of coordinate operators in the field ϕ, see [161, 162]. However, equation (7.73) is not a differential equation, so we cannot rely on the mathematical theorems about the completeness of the solutions, and no one guarantees that we can find all the solutions in this way. More fundamentally, we have not defined the vector space on which operators $\hat{p}_a$ and $\hat{x}^\mu$ act. We do this by working in a concrete representation of the $\mathfrak{sl}(2,\mathbb{R})$ algebra. Recall that we made the classifications of unitary irreducible representations of this algebra in chapter 5. We take that the operators act on a discrete series representation T_l^-, with l being a negative half-integer. One way to realize this representation is by acting on functions of a single real variable $\xi > 0$, see Appendix A.2

for details. We can then write down a concrete differential operators that correspond to the fuzzy AdS_2 coordinates and momenta as

$$\hat{z} = \bar{k}\xi\,, \qquad \hat{p}_z = \xi\partial_\xi + l + 1\,, \tag{7.74}$$

$$\hat{t} = -i\bar{k}\left(\xi\partial_\xi + l + 1\right)\,, \qquad \hat{p}_t = -i\xi\,. \tag{7.75}$$

Therefore, our construction of fuzzy AdS_2 spacetime depends on two numbers: $\bar{k}$ and l. Similarly, we take the generators in the three-dimensional case to act on a vector space $\mathcal{H}_l \oplus \bar{\mathcal{H}}_l$ of $T_l^- \oplus \bar{T}_l^-$ representation. Operators and momenta act on functions $f(\xi, \bar{\xi})$ of two variables, and the exact form of operators can be found using the relations

$$H = \xi\partial_\xi + l + 1\,, \quad E_+ = -i\xi\,, \qquad \bar{H} = \bar{\xi}\partial_{\bar{\xi}} + l + 1\,, \quad \bar{E}_+ = -i\bar{\xi}\,. \tag{7.76}$$

Let us focus on a two-dimensional example. The algebra of functions is given by

$$\mathcal{A} = \mathrm{End}(\mathcal{H}_l) \cong T_l^- \otimes (T_l^-)^* \cong T_l^- \otimes (T_l^-)^B\,, \tag{7.77}$$

see Appendix A.2 for the last equation. We shall label the variable ξ from the first factor in (7.77) as ξ_L, and the other one as ξ_R. Therefore, the algebra $\mathcal{A}$ can be thought of as made of two-variable functions, on which the momenta act as

$$\mathrm{ad}_{\hat{p}_z} = \xi_L\partial_{\xi_L} + \xi_R\partial_{\xi_R} + 2l + 2\,, \qquad \mathrm{ad}_{\hat{p}_t} = -i\left(\xi_L - \xi_R\right)\,. \tag{7.78}$$

We make the following change of variables

$$\xi_L = \rho\sin\varphi\,, \quad \xi_R = \rho\cos\varphi\,, \qquad \rho \in (0,\infty)\,, \quad \varphi \in \left(0, \frac{\pi}{2}\right)\,, \tag{7.79}$$

where the range of φ coorinate will be important later. Let us denote

$$\langle\xi_L|\phi|\xi_R\rangle = \langle\rho\sin\varphi|\phi|\rho\cos\varphi\rangle \equiv \phi(\rho,\varphi)\,. \tag{7.80}$$

By taking $\phi(\rho,\varphi) = \rho^{-2l-2}g(\rho,\varphi)$, NC KG equation (7.73) can then be written as

$$\rho^2\frac{\partial^2 g(\rho,\varphi)}{\partial\rho^2} + \left(2\rho^2\cos^2\left(\varphi + \frac{\pi}{4}\right) - M^2\right)g(\rho,\varphi) = 0\,. \tag{7.81}$$

As this equation does not contain any derivatives with respect to the φ variable, we can take the modes of the form $g(\rho,\varphi) = f(\rho)\delta(\varphi - \varphi_0)$. If we define

$$\beta = \sqrt{2}\cos\left(\varphi_0 + \frac{\pi}{4}\right)\,, \qquad \beta \in (-1,1)\,, \tag{7.82}$$

the equation (7.81) becomes

$$\rho^2\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \beta^2\right)f = M^2 f\,. \tag{7.83}$$

The solutions of this equation are given in terms of Bessel's functions, and choosing the one that is regular for $\rho \to 0$, we get

$$f(\rho) = c_\beta\sqrt{\rho}\,J_\nu\left(\beta\rho\right)\,, \tag{7.84}$$

where $\nu = \sqrt{\frac{1}{4} + m^2}$. The three-dimensional case can be handled in the same manner, see [149]. First, in representation, we work with functions $\phi(\xi_L, \xi_R, \bar{\xi}_L, \bar{\xi}_R)$, and the momenta are given by

$$\mathrm{ad}_{\hat{p}_z} = \xi_L\partial_{\xi_L} + \xi_R\partial_{\xi_R} + \bar{\xi}_L\partial_{\bar{\xi}_L} + \bar{\xi}_R\partial_{\bar{\xi}_R} + 4l + 4\,, \tag{7.85}$$

$$\mathrm{ad}_{\hat{p}_t} = -i\left(\xi_L - \xi_R - \bar{\xi}_L + \bar{\xi}_R\right)\,, \qquad \mathrm{ad}_{\hat{p}_x} = -i\left(\xi_L - \xi_R + \bar{\xi}_L - \bar{\xi}_R\right)\,. \tag{7.86}$$

We make the change of variables as

$$\xi_L = \chi_L\, e^{\zeta_L}\,, \quad \bar{\xi}_L = \chi_L\, e^{-\zeta_L}\,, \qquad \xi_R = \chi_R\, e^{\zeta_R}\,, \quad \bar{\xi}_R = \chi_R\, e^{-\zeta_R}\,, \tag{7.87}$$

with $\chi_L, \chi_R \in (0,\infty)$ and $\zeta_L, \zeta_R \in (-\infty,\infty)$. Furthermore, we introduce variables

$$\rho = \sqrt{\chi_L^2+\chi_R^2}\,, \qquad \tan\varphi = \frac{\chi_R}{\chi_L}\,, \qquad \zeta_\pm = \frac{1}{2}\left(\zeta_L \pm \zeta_R\right), \tag{7.88}$$

such that $\rho \in (0,\infty)$, $\varphi \in (0,\frac{\pi}{2})$, $\zeta_\pm \in (-\infty,\infty)$. Similar to the two-dimensional case, we take $\phi = \rho^{-\frac{8l+7}{2}}\, g(\rho,\varphi,\zeta_+,\zeta_-)$, such that the KG equation is

$$\left(\rho^2\partial_\rho^2 - 4\rho^2\big(1-\cosh 2\zeta_-\sin 2\varphi\big) - \frac{3}{4}\right) g = M^2 g\ . \tag{7.89}$$

Again, this equation does not contain derivatives of variables other than ρ, so we take modes

$$g(\rho,\varphi,\zeta_+,\zeta_-) = f(\rho)\,\delta(\varphi-\varphi_0)\,\delta(\zeta_+-\zeta_{+0})\,\delta(\zeta_--\zeta_{-0})\,. \tag{7.90}$$

Solving the equation for $f(\rho)$, we obtain the mode that does not diverge for $\rho\to 0$ as

$$\phi_\gamma = c_\gamma\,\rho^{-4l-3}\,J_\nu(2\gamma\rho)\,\delta(\varphi-\varphi_0)\,\delta(\zeta_+-\zeta_{+0})\,\delta(\zeta_--\zeta_{-0})\,. \tag{7.91}$$

We defined $\nu=\sqrt{1+m^2}$ and

$$\gamma^2 = \cosh 2\zeta_{-0}\,\sin 2\varphi_0 - 1\ . \tag{7.92}$$

We assume that $\gamma^2 > 0$, which restricts the range of parameters in the following sections. Note that there is a considerable difference between the two-dimensional case and the three-dimensional case. In the former one, NC modes depend on only one quantum number, labeled β. However, in the three-dimensional case, NC modes depend on three quantum numbers, while the commutative modes depend only on two, see section 3.1.

7.4.3 Semi-classical states: concept of boundary

We saw in chapter 3 that the boundary limit of the AdS spacetime in Poincaré coordinates (1.32) is $z\to 0$. How should we introduce this boundary limit on the fuzzy AdS spacetime, where all the coordinates are given by operators acting on a Hilbert space of the T_l^- representation? We introduce a set of *semiclassical* states, labeled by the classical points of AdS spacetime. We will compute all the observables as expectation values in those semiclassical states. Using the definitions from Appendix A.2, we define semiclassical states in the case of the fuzzy AdS_2 as

$$|\lambda, c\rangle \equiv \lambda^{-\hat{p}_z}\, e^{-c\hat{p}_t}\,|\Psi_0\rangle\ . \tag{7.93}$$

In this formula, , $|\Psi_0\rangle$ is the lowest-weight vector from the T_l^- representation, defined as $\tilde{E}_-|\Psi_0\rangle = 0$, $\tilde{H}|\Psi_0\rangle = -l|\Psi_0\rangle$. Explicitly, it is given as

$$\Psi_0(\xi) = \frac{\xi^{-2l-1}e^{-\xi}}{2^{2l+1/2}\sqrt{\pi\Gamma(-2l)}} \equiv N\xi^{-2l-1}e^{-\xi}\ . \tag{7.94}$$

It is straightforward to check that

$$\langle\lambda, c\,|\hat{z}|\lambda, c\rangle = \lambda\langle\Psi_0|\hat{z}|\Psi_0\rangle = -l\hbar\lambda\,, \tag{7.95}$$

$$\langle\lambda, c\,|\hat{t}|\lambda, c\rangle = \langle\Psi_0|\hat{t}+c\hat{z}|\Psi_0\rangle = -l\hbar c\ . \tag{7.96}$$

Semiclassical state $|\lambda, c\rangle$ corresponds to the classical point obtained by translation

$$\lambda^{\mathsf{e}_z}\, e^{c\mathsf{e}_t} \cdot (z_0, t_0) = (\lambda z_0, t_0 + c z_0) = (-l\bar{k}\lambda, -l\bar{k}c) \ , \tag{7.97}$$

where $(z_0, t_0) = (-l\bar{k}, 0)$. From (7.95), we see that the boundary limit can be defined by taking $\lambda \to 0$. At this point, we note that the semiclassical states in two dimensions resolve the unity,

$$\int_{-\infty}^{\infty} \mathrm{d}c \int_{0}^{\infty} \frac{\mathrm{d}\lambda}{\lambda}\, |\lambda, c\rangle\langle\lambda, c| \equiv P = -\frac{4\pi}{2l+1}\, I \ . \tag{7.98}$$

Also, for concrete computations that follow, we shall use relation

$$\langle\xi|\lambda, c\rangle = N\,\lambda^l\, \xi^{-2l-1}\, e^{-\frac{1-ic}{\lambda}\xi}\,, \qquad N = \left(2\pi\, 2^{4l}\,\Gamma(-2l)\right)^{-1/2} \ . \tag{7.99}$$

Most of this construction generalizes to three dimensions. Semiclassical states are similarly defined as

$$|\lambda, b, c\rangle = \lambda^{-\hat{p}_z} e^{-b\hat{p}_x} e^{-c\hat{p}_t} |\Psi_0 \otimes \bar{\Psi}_0\rangle\,. \tag{7.100}$$

We have

$$\langle\xi, \bar{\xi}|\lambda, b, c\rangle = N^2\lambda^{2l}\left(\xi\bar{\xi}\right)^{-2l-1} e^{\frac{i}{\lambda}\left(\xi(b+c+i)+\bar{\xi}(b-c+i)\right)}\,, \tag{7.101}$$

and the expectation values of the coordinates turn out to be given as

$$\langle\lambda, b, c\,|\hat{z}|\lambda, b, c\rangle = \frac{\Gamma\left(\frac{1}{2}-2l\right)^2}{\Gamma(-2l)^2}\,\bar{k}\lambda\, \sim -2l\bar{k}\lambda\ , \tag{7.102}$$

$$\langle\lambda, b, c\,|\hat{t}|\lambda, b, c\rangle = \frac{\Gamma\left(\frac{1}{2}-2l\right)^2}{\Gamma(-2l)^2}\,\bar{k}c\, \sim -2l\bar{k}c\ , \tag{7.103}$$

$$\langle\lambda, b, c\,|\hat{x}|\lambda, b, c\rangle = \frac{\Gamma\left(\frac{1}{2}-2l\right)^2}{\Gamma(-2l)^2}\,\bar{k}b\, \sim -2l\bar{k}b\ . \tag{7.104}$$

The approximations made in previous formulas hold for large l, and we shall see in the next section that this limit, together with $\bar{k} \to 0$ limit, represents the classical limit of the NC bulk. Lastly, note that in three dimensions, it is necessary to introduce another generator $H - \bar{H}$ in (7.100) to get the resolution of the identity similar to the two-dimensional case.

7.4.4 Asymptotic behavior and classical limit

We are in a position to compute the expectation value of the NC modes in semiclassical states. On fuzzy AdS_2, we have

$$\langle\phi_\beta\rangle = \left(2^{2l+1}\pi\right)^2 \int_0^\infty\!\!\int_0^\infty (\xi_L\xi_R)^{2l+1} \mathrm{d}\xi_L\, \mathrm{d}\xi_R\ \phi_\beta(\xi_L, \xi_R)\langle\xi_L|\lambda, c\rangle\langle\xi_R|\lambda, c\rangle^* \ . \tag{7.105}$$

Using the explicit formula (7.99) and integrating over the delta function, we have

$$\langle\phi_\beta\rangle = \frac{2\pi\,\lambda^{2l} c_\beta}{\Gamma(-2l)} \int_0^\infty \mathrm{d}\rho\,\rho^{-2l-\frac{1}{2}}\, J_\nu(\beta\rho)\ e^{-\frac{\rho}{\lambda}(\sin\varphi_0+\cos\varphi_0)+\frac{ic\rho}{\lambda}(\sin\varphi_0-\cos\varphi_0)} \ . \tag{7.106}$$

Interestingly, this integral can be evaluated using [91]

$$\int_0^\infty \mathrm{d}x\, x^{\mu-1} e^{-\alpha x} J_\nu(\beta x) = \frac{\left(\frac{\beta}{2\alpha}\right)^\nu \Gamma(\nu+\mu)}{\alpha^\mu \Gamma(\nu+1)}\ {}_2F_1\left(\frac{\nu+\mu}{2}, \frac{\nu+\mu+1}{2}; \nu+1; -\frac{\beta^2}{\alpha^2}\right), \tag{7.107}$$

assuming

$$\mathrm{Re}(\alpha+i\beta) > 0\,, \qquad \mathrm{Re}(\alpha-i\beta) > 0\,, \qquad \mathrm{Re}(\mu+\nu) > 0\ . \tag{7.108}$$

A concrete computation gives

$$\langle\phi_\beta\rangle = \frac{2\pi\,\lambda^{2l}\,c_\beta}{\Gamma(-2l)}\,\frac{\left(\frac{\beta}{2}\right)^\nu \Gamma(\nu-2l+\frac{1}{2})}{\alpha^{\nu-2l+1/2}\,\Gamma(\nu+1)}\ {}_2F_1\left(\frac{\nu-2l+\frac{1}{2}}{2}, \frac{\nu-2l+\frac{3}{2}}{2}; \nu+1; -\frac{\beta^2}{\alpha^2}\right)\ , \tag{7.109}$$

where we introduced

$$\mu = -2l + \frac{1}{2}\,, \qquad \alpha = \frac{\sqrt{2-\beta^2}+ic\beta}{\lambda}\ . \tag{7.110}$$

There are two important properties of this expression that we now discuss. First, let us take the boundary limit $\lambda \to 0$. Using the property of hypergeometric function that ${}_2F_1(\ldots;0) = 1$, we can obtain that in this limit, mode (7.109) behave as $\sim \langle z\rangle^{\frac{1}{2}+\nu} = \langle z\rangle^\Delta$, where we recognized the definition from (3.22). Importantly, this coincides with the behaviour of the commutative modes, and enables us to later use the extrapolate dictionary (3.23) to obtain the boundary two-point function. Moreover, if we had chosen the other solution to (7.83), proportional to $Y_\nu(\beta\rho)$, and therefore diverges for small ρ, we would have obtained the small λ behavior in the form of $\sim \langle z\rangle^{1-\Delta}$. This is precisely the near-boundary behavior of the nonnormalizable solution of the commutative equation (3.5).

An interesting property of the obtained modes is that in the limit $|l| \to \infty$, they obtain the form of the commutative modes (3.10). Note the following limit [163]

$$J_\nu(\tilde z) = \lim_{a,b\to\infty} \frac{\left(\frac{\tilde z}{2}\right)^\nu}{\Gamma(\nu+1)}\ {}_2F_1\left(a, b; \nu+1; -\frac{\tilde z^2}{4ab}\right)\ . \tag{7.111}$$

In order to make the connection with (7.109), we take

$$a = -l + \frac{\nu+\frac{1}{2}}{2}\,, \qquad b = -l + \frac{\nu+\frac{3}{2}}{2}\,, \tag{7.112}$$

resulting in

$$\tilde z^2 = 4\left(-l + \frac{\nu+\frac{1}{2}}{2}\right)\left(-l + \frac{\nu+\frac{3}{2}}{2}\right)\frac{\beta^2}{\left(\sqrt{2-\beta^2}+ic\beta\right)^2}\,\frac{\langle\hat z\rangle^2}{l^2\hbar^2}\ . \tag{7.113}$$

This implies that in $|l| \to \infty$ limit, both a and b are infinite, but the $\tilde z$ is finite

$$\tilde z \sim \frac{2\beta\langle\hat z\rangle}{\hbar\left(\sqrt{2-\beta^2} - i\frac{\langle\hat t\rangle}{l\hbar}\beta\right)} \sim \frac{2\beta\langle\hat z\rangle}{\hbar\sqrt{2-\beta^2}}\ . \tag{7.114}$$

Combining previous formulas, we have

$$\begin{aligned}\langle\phi_\beta\rangle &\sim 2\pi c_\beta \sqrt{\frac{2\langle\hat z\rangle}{\hbar}}\left(\sqrt{2-\beta^2} - i\frac{\beta\langle\hat t\rangle}{l\hbar}\right)^{2l-1/2} J_\nu\left(\frac{2\beta\langle\hat z\rangle}{\hbar\sqrt{2-\beta^2}}\right)\\ &\sim 2\pi c_\beta\left(2-\beta^2\right)^{l-\frac{1}{4}} e^{-i\frac{2\beta\langle\hat t\rangle}{\hbar\sqrt{2-\beta^2}}}\sqrt{\frac{2\langle\hat z\rangle}{\hbar}}\ J_\nu\left(\frac{2\beta\langle\hat z\rangle}{\hbar\sqrt{2-\beta^2}}\right)\,,\end{aligned} \tag{7.115}$$

with the last equation following from the well-known expression $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$. If we introduce

$$\omega = \frac{2\beta}{\bar{k}\sqrt{2-\beta^2}}\,, \qquad \omega \in \left(\frac{-2}{\bar{k}}, \frac{2}{\bar{k}}\right)\,, \tag{7.116}$$

the $|l| \to \infty$ limit of NC modes (7.115) can be rewritten as

$$\langle \phi_\beta \rangle \sim \frac{4\pi c_\beta}{\sqrt{\bar{k}}}\left(\frac{8}{4+\bar{k}^2\omega^2}\right)^{l-\frac{1}{4}} \frac{1}{\sqrt{2}}\, e^{-i\omega\langle \hat{t}\rangle}\sqrt{\langle \hat{z}\rangle}\; J_\nu\left(\omega\langle \hat{z}\rangle\right)\,. \tag{7.117}$$

If we choose the normalisation constant c_β as

$$c_\beta = \frac{\sqrt{\bar{k}}}{4\pi}\left(\frac{8}{4+\bar{k}^2\omega^2}\right)^{-l+\frac{1}{4}}\,, \tag{7.118}$$

the obtained result matches exactly with (3.10). Furthermore, we see that the noncommutativity introduces a natural cut-off on the frequencies $\omega_{\text{cut-off}} = \frac{2}{\bar{k}}$. In order to restore a fully classical limit, we should send this cut-off to infinity, implying $\bar{k} \to 0$. Therefore, we conclude that the classical limit is given by

$$\lim_{\bar{k}\to 0}\lim_{l\to -\infty} \qquad \text{or} \qquad |l| \gg \frac{1}{\bar{k}} \gg 1\,. \tag{7.119}$$

A similar conclusion holds in three dimensions. The expectation value of NC modes is given by

$$\begin{aligned}\langle \phi_\gamma \rangle = {} & (2^{2l+1}\pi)^4 \int_0^\infty\!\!\int_0^\infty\!\!\int_0^\infty\!\!\int_0^\infty (\xi_L\bar{\xi}_L\xi_R\bar{\xi}_R)^{2l+1}\, \mathrm{d}\xi_L\, \mathrm{d}\bar{\xi}_L\, \mathrm{d}\xi_R\, \mathrm{d}\bar{\xi}_R \\ & \times \phi_\gamma(\xi_L,\bar{\xi}_L,\xi_R,\bar{\xi}_R)\langle \xi_L,\bar{\xi}_L|\lambda,b,c\rangle\langle \xi_R,\bar{\xi}_R|\lambda,b,c\rangle^* \\ = {} & 2N^4\left(2^{2l+1}\pi\right)^4 c_\gamma \iiiint \mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\zeta_+\,\mathrm{d}\zeta_-\; \lambda^{4l}\rho^{-4l}\sin(2\varphi)\, e^{-\frac{2\rho}{\lambda}f_\lambda + \frac{2i\rho}{\lambda}(bf_b+cf_c)} \\ & \times J_\nu(2\gamma\rho)\delta(\varphi-\varphi_0)\,\delta(\zeta_+-\zeta_{+0})\,\delta(\zeta_--\zeta_{-0})\,. \end{aligned} \tag{7.120}$$

In the last formula, we introduced labels

$$f_\lambda = \cos\varphi\cosh\zeta_L + \sin\varphi\cosh\zeta_R\,, \tag{7.121}$$
$$f_b = \cos\varphi\cosh\zeta_L - \sin\varphi\cosh\zeta_R\,, \tag{7.122}$$
$$f_c = \cos\varphi\sinh\zeta_L - \sin\varphi\sinh\zeta_R\,. \tag{7.123}$$

After performing the integrals of delta functions, we are left with

$$\langle \phi_\gamma \rangle = 2N^4\left(2^{2l+1}\pi\right)^4 c_\gamma\lambda^{4l}\sin(2\varphi_0)\int \mathrm{d}\rho\; \rho^{-4l}\, e^{-\frac{2\rho}{\lambda}f_{\lambda 0} + \frac{2i\rho}{\lambda}(bf_{b0}+cf_{c0})} J_\nu(2\gamma\rho)\,. \tag{7.124}$$

At this stage, we can again use the integral (7.107), with

$$\mu = 1-4l\,, \qquad \beta = 2\gamma\,, \qquad \alpha = \frac{2f_{\lambda 0} - 2i(bf_{b0}+cf_{c0})}{\lambda}\,, \tag{7.125}$$

to get

$$\begin{aligned}\langle \phi_\gamma \rangle = {} & 2^{8l+5}\pi^4 N^4 c_\gamma \lambda^{4l}\sin 2\varphi_0\, \frac{\Gamma(\nu+1-4l)}{\Gamma(\nu+1)}\, \frac{\gamma^\nu}{\alpha^{\nu+1-4l}} \\ & \times {}_2F_1\left(\frac{\nu+1-4l}{2}, \frac{\nu+2-4l}{2}; \nu+1; -\frac{4\gamma^2}{\alpha^2}\right)\,. \end{aligned} \tag{7.126}$$

Again, we have the same near-boundary behaviour as in the commutative case: modes (7.126) behave as $\sim \langle\hat{z}\rangle$ for $\lambda \to 0$. Furthermore, we can again show that the limit (7.119) corresponds to the classical limit. First, note that we can rewrite α as

$$\alpha = \frac{-4l\hbar f_{\lambda 0}}{\langle\hat{z}\rangle}\left(1+\frac{i}{2l\hbar}\left(\frac{f_{b0}}{f_{\lambda 0}}\langle\hat{x}\rangle+\frac{f_{c0}}{f_{\lambda 0}}\langle\hat{t}\rangle\right)\right) \equiv \frac{-4l\hbar f_{\lambda 0}}{\langle\hat{z}\rangle}\left(1-\frac{i\omega\langle\hat{t}\rangle}{4l}+\frac{ik\langle\hat{x}\rangle}{4l}\right) , \tag{7.127}$$

where we introduced

$$\omega = -\frac{2}{\hbar}\frac{f_{c0}}{f_{\lambda 0}} , \qquad k = \frac{2}{\hbar}\frac{f_{b0}}{f_{\lambda 0}} . \tag{7.128}$$

We can again use formula (7.111) with

$$a = -2l + \frac{\nu+1}{2} , \qquad b = -2l + \frac{\nu+2}{2}, \qquad \tilde{z}^2 = 16ab\frac{\gamma^2}{\alpha^2} . \tag{7.129}$$

Finally, we note that we have

$$\lim_{l\to-\infty} \tilde{z} = \frac{2\gamma\langle\hat{z}\rangle}{\hbar f_{\lambda 0}} = \sqrt{\omega^2-k^2}\,\langle\hat{z}\rangle , \tag{7.130}$$

resulting in

$$\begin{aligned}\langle\phi_\gamma\rangle = \frac{2^{4l+3}\pi^2}{\Gamma(-2l)^2}\, c_\gamma \,\sin 2\varphi_0\, \frac{\Gamma(\nu+1-4l)}{(-4l)^{\nu+1}}\left(1-\frac{i\omega\langle\hat{t}\rangle}{4l}+\frac{ik\langle\hat{x}\rangle}{4l}\right)^{4l-\nu-1} \\ \times\left(\frac{1}{f_{\lambda 0}}\right)^{1-4l}\hbar^{-1}\langle\hat{z}\rangle J_\nu\left(\sqrt{\omega^2-k^2}\,\langle\hat{z}\rangle\right) .\end{aligned} \tag{7.131}$$

After recognizing the definition of the exponential function as before, we get the promised form of classical AdS_3 modes

$$\langle\phi_\gamma\rangle = 2^{\frac{7}{2}}\pi^2(-l)^{\frac{1}{2}}\hbar^{-1}c_\gamma\,(f_{\lambda 0})^{4l-1}\sin 2\varphi_0\,\frac{1}{\sqrt{4\pi}}\,e^{-i\omega\langle\hat{t}\rangle+ik\langle\hat{x}\rangle}\langle\hat{z}\rangle\, J_\nu\left(\sqrt{\omega^2-k^2}\,\langle\hat{z}\rangle\right) . \tag{7.132}$$

Two important comments are in order. First, both ω and k are bounded, as before. Second, once we computed the expectation values of NC modes in semiclassical states, the obtained result depends only on two quantum numbers ω and k, up to a normalization factor. Later, we will choose the constant c_γ such that we make the simplest possible choice that gives the commutative result in the classical limit.

7.4.5 QFT and boundary two-point function from extrapolate dictionary

So far, we have had quantum spacetime and a classical scalar field on top of it. We should next define the quantum field on the quantum spacetime. This is a nontrivial task, and we have to make some assumptions that we are not able to derive from the first principles. First, we will assume that there exists a Fock space, with a unique vacuum $|0\rangle$, and creation/annihilation operators $a_{\omega,k,s}$. The NC field is expanded as

$$\phi = \iiint \mathrm{d}\omega\,\mathrm{d}k\,\mathrm{d}s\left(\phi_{\omega,k,s}a_{\omega,k,s}+\phi^*_{\omega,k,s}a^\dagger_{\omega,k,s}\right) , \tag{7.133}$$

where s stands for any possible quantum number different from ω and k. Second, we will assume that the creation and annihilation operators satisfy the same algebra as in the commutative case

$$[a_{\omega,k,s}, a^\dagger_{\omega',k',s'}] = \delta(\omega - \omega')\,\delta(k - k')\,\delta(s - s')\,, \qquad [a, a] = [a^\dagger, a^\dagger] = 0\ . \tag{7.134}$$

Standardly, this is derived from the canonical quantization of the scalar field, which induces the equal-time commutation relations. Unfortunately, on NC spacetime, it is not possible to define equal-time commutational relations, as time is itself an operator. Furthermore, in our approach, we have only defined the scalar field $\hat{\phi}$ and not its momentum, and as we are lacking the Lagrangian of our theory, it is hard to define this quantity. Next, we assume that the annihilation operators annihilate the vacuum state

$$a_{\omega,k,s}|0\rangle = 0\ . \tag{7.135}$$

In a nutshell, we assume that the standard quantization in terms of Fock space holds. It is then easy to compute the two-point function as

$$\begin{aligned} G_2 = \langle \Phi \otimes \Phi \rangle &= \iint \mathrm{d}\Lambda\, \mathrm{d}\Lambda'\, \langle 0|(\phi_\Lambda a_\Lambda + \phi^*_\Lambda a^\dagger_\Lambda) \otimes (\phi_{\Lambda'} a_{\Lambda'} + \phi^*_{\Lambda'} a^\dagger_{\Lambda'})|0\rangle \\ &= \iint \mathrm{d}\Lambda\, \mathrm{d}\Lambda'\, \langle 0|\phi_\Lambda a_\Lambda \otimes \phi^*_{\Lambda'} a^\dagger_{\Lambda'}|0\rangle = \iint \mathrm{d}\Lambda\, \mathrm{d}\Lambda'\, \phi_\Lambda \otimes \phi^*_{\Lambda'}\, \langle 0|[a_\Lambda, a^\dagger_{\Lambda'}]|0\rangle\ . \end{aligned} \tag{7.136}$$

where we used a symbol Λ for all quantum numbers. Using (7.134), we get

$$\langle \Phi \otimes \Phi \rangle = \iiint \mathrm{d}\omega\, \mathrm{d}k\, \mathrm{d}s\, \langle \phi_{\omega,k,s} \otimes \phi^*_{\omega,k,s} \rangle\ . \tag{7.137}$$

Field (7.133) is both an operator on a Fock space of quantum field theory and an operator on a fuzzy spacetime. After calculating the expectation value in the Fock vacuum, the obtained correlation function is $\mathrm{End}(\mathcal{H})$-valued. We further wish to evaluate the expectation value of this two-point function in the semiclassical states, obtaining expressions that are ordinary functions in the form of

$$G_n(\mathtt{x}_1, \ldots, \mathtt{x}_n) = \langle \mathtt{x}_1, \ldots, \mathtt{x}_n | \langle \Phi \otimes \cdots \otimes \Phi \rangle | \mathtt{x}_1, \ldots, \mathtt{x}_n \rangle\ . \tag{7.138}$$

In this formula, $|\mathtt{x}_1, \ldots, \mathtt{x}_n\rangle$ stands for the semi-classical state corresponding to the classical point $(\mathtt{x}_1, \ldots, \mathtt{x}_n)$, obtained as a tensor product of semi-classical states $|x_i\rangle$. We can finally compare those functions to the commutative correlation functions and see how the NC effects modify the standard CFT result for the two-point function. In our example, we should compute

$$G_2(\mathtt{x}_1, \mathtt{x}_2) = \iiint \mathrm{d}\omega\, \mathrm{d}k\, \mathrm{d}s\, \langle \mathtt{x}_1 | \phi_{\omega,k,s} | \mathtt{x}_1 \rangle \langle \mathtt{x}_2 | \phi_{\omega,k,s} | \mathtt{x}_2 \rangle^*\ . \tag{7.139}$$

Let us perform a concrete computation in two dimensions. Using modes (7.109), we have

$$G_2(\lambda_i, c_i) = \int \mathrm{d}\omega\, \langle \lambda_1, c_1 | \phi_\beta | \lambda_1, c_1 \rangle\, \langle \lambda_2, c_2 | \phi^*_\beta | \lambda_2, c_2 \rangle\ . \tag{7.140}$$

It is important to once again stress the range of quantum number ω, as it is different form the standard commutative case: contrary to the standard interval $\omega \in (-\infty, +\infty)$, we have $\omega \in (-\frac{2}{k}, \frac{2}{k})$. This cut-off makes the integral manifestly finite. Furthermore, in the commutative theory, we should integrate only over positive ω, corresponding to the positive-frequency modes; we do the same on the fuzzy spacetime. At this point, we also stress that we decided to integrate over ω instead of over β, as we know that integration over ω, with the normalization chosen as in the last section, provides us with the correct semi-classical limit. We are not aware of any more systematic way to define the correlation functions. Plugging in the exact form of modes, we obtain

$$G_2(\lambda_i, c_i) = (2\pi)^2 \frac{\Gamma(\nu - 2l + \frac{1}{2})^2}{\Gamma(-2l)^2\Gamma(\nu+1)^2} (\lambda_1\lambda_2)^{2l} \times \int_0^{2\hbar^{-1}} d\omega\, c_\beta^2 \frac{\left(\frac{\beta}{2}\right)^{2\nu}}{(\alpha_1\alpha_2^*)^{\nu-2l+1/2}} \prod_{i=1}^{2} {}_2F_1\left(\frac{\nu-2l+\frac{1}{2}}{2}, \frac{\nu-2l+\frac{3}{2}}{2}; \nu+1; -\frac{\beta^2}{\alpha_{(i)}^2}\right). \tag{7.141}$$

Let us denote $\alpha_{(1)} = \alpha_1$ and $\alpha_{(2)} = \alpha_2^*$. Even though we cannot compute this integral, the boundary limit obtained using (3.23) is much simpler. In the limit $\langle\hat{z}_i\rangle \to 0$, we have $\alpha_i \to \infty$, so that the last argument of the hypergeometric function tends to zero. In this limit, the hypergeometric function itself tends to unity, so that we have

$$\tilde{G}_{\partial\partial}(\lambda_i, c_i) = 4\pi^2 \frac{\Gamma(\nu - 2l + \frac{1}{2})^2}{\Gamma(-2l)^2\Gamma(\nu+1)^2} (\lambda_1\lambda_2)^{2l} \int_0^{2\hbar^{-1}} d\omega\, c_\beta^2 \frac{\left(\frac{\beta}{2}\right)^{2\nu}}{(\alpha_1\alpha_2^*)^{\nu-2l+1/2}}\,. \tag{7.142}$$

This is not yet the correlation function of two operators at the boundary, as we have to strip off the factors $G_{\partial\partial} = \langle\hat{z}_1\rangle^{-\Delta}\langle\hat{z}_2\rangle^{-\Delta}\tilde{G}_{\partial\partial}$, as indicated by (3.23). This results in

$$\tilde{G}_{\partial\partial} = 4\pi^2 \frac{\Gamma(\nu - 2l + \frac{1}{2})^2}{\Gamma(-2l)^2\Gamma(\nu+1)^2} \left(\frac{\langle\hat{z}_1\rangle}{-l\hbar}\right)^{\nu+1/2} \left(\frac{\langle\hat{z}_2\rangle}{-l\hbar}\right)^{\nu+1/2} \times \int_0^{2\hbar^{-1}} d\omega\, c_\beta^2\, (2-\beta^2)^{2l-1/2} \left(1 - \frac{i\omega\langle\hat{t}_1\rangle}{2l}\right)^{2l-\nu-1/2} \left(1 + \frac{i\omega\langle\hat{t}_2\rangle}{2l}\right)^{2l-\nu-1/2} \left(\frac{\hbar\omega}{4}\right)^{2\nu}. \tag{7.143}$$

Finally, by taking the normalization (7.118), the expression (7.143) simplifies to

$$G_{\partial\partial} = \frac{\Gamma(\Delta - 2l)^2}{4^{2\Delta}\Gamma(-2l)^2\Gamma\left(\Delta+\frac{1}{2}\right)^2} \left(\frac{1}{-l}\right)^{2\Delta} \int_0^{2\hbar^{-1}} d\omega \left(1 - \frac{i\omega\langle\hat{t}_1\rangle}{2l}\right)^{2l-\Delta} \left(1 + \frac{i\omega\langle\hat{t}_2\rangle}{2l}\right)^{2l-\Delta} \omega^{2\Delta-1}\,. \tag{7.144}$$

Before computing this integral, let us pause for a moment and check what happens in the commutative limit. Assuming that the limit procedure commutes with the integral, the limit $l \to -\infty$ provides us with the expression of the form $\omega^{2\Delta-1}e^{-i\omega\langle\hat{t}_{12}\rangle}$ under the integral. In addition, limit $\hbar \to 0$ leads to the standard integration bounds $\omega \in (0, +\infty)$. If $\Delta < \frac{1}{2}$, the integral results precisely in the desired conformal two-point function

$$G_{\partial\partial}^{\text{comm}} \equiv \frac{4^{-\Delta}}{\Gamma(\Delta+\frac{1}{2})^2} \int_0^{\infty} d\omega\, \omega^{2\Delta-1} e^{-i\omega\langle\hat{t}_{12}\rangle} = C_\Delta \langle i\hat{t}_{12}\rangle^{-2\Delta}\,. \tag{7.145}$$

However, the integral formally diverges when $\Delta \geq \frac{1}{2}$ (which is a necessary condition for the BF bound to be satisfied), and we define the two-point function in this range as an analytic continuation of the $\Delta < \frac{1}{2}$ result. We will take the same logic for the fuzzy two-point function. Having said this, we now observe that the integral in (7.144) can be analytically expressed in terms of the Appell $F1$ function as

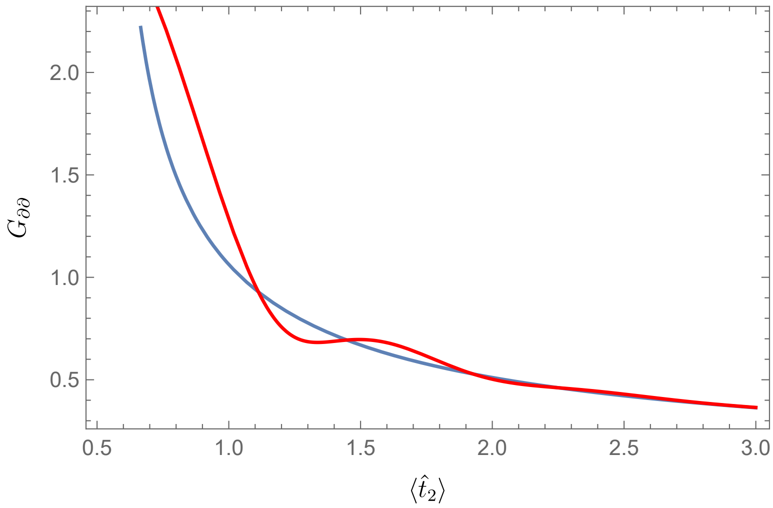

Figure 7.1: The fuzzy two-point function at the boundary (red) vs. the commutative (conformal) two-point function (blue). In this plot, numerical values of constants are: $l = -30$, $\bar{k} = 1/4$, $\Delta = 1/3$, $t_1 = 1/2$.

$$G_{\partial\partial} = \frac{2^{-1-2\Delta}\Gamma(\Delta-2l)^2}{\Gamma(-2l)^2\Gamma\left(\Delta+\frac{1}{2}\right)^2\Delta}\left(\frac{1}{-l}\right)^{2\Delta}\bar{k}^{-2\Delta}F_1\left(2\Delta,\Delta-2l,\Delta-2l,2\Delta+1;\frac{i\langle\hat{t}_1\rangle}{l\bar{k}},\frac{-i\langle\hat{t}_2\rangle}{l\bar{k}}\right). \tag{7.146}$$

We plot this function in Figure 7.1. Note that, while we already discussed the commutative limit before computing the integral, we can explicitly compute it here, using the results from [164]. First, we have

$$\lim_{l\to-\infty} G_{\partial\partial} = \frac{\bar{k}^{-2\Delta}}{2\Gamma(\Delta+\frac{1}{2})^2\,\Delta}\,{}_1F_1\left(2\Delta;2\Delta+1;\frac{-2i\langle\hat{t}_{12}\rangle}{\bar{k}}\right), \tag{7.147}$$

where $\langle\hat{t}_{12}\rangle = \langle\hat{t}_1\rangle - \langle\hat{t}_2\rangle$. This two-point function is shown in Figure 7.2. Furthermore, by taking the $\bar{k}\to 0$ limit, we end up with

$$\lim_{\bar{k}\to 0}\lim_{l\to-\infty} G_{\partial\partial} = \frac{\bar{k}^{-2\Delta}\Gamma(2\Delta+1)}{2\Gamma(\Delta+\frac{1}{2})^2\,\Delta}\left(\frac{2i\langle\hat{t}_{12}\rangle}{\bar{k}}\right)^{-2\Delta} = C_\Delta\,\langle i\hat{t}_{12}\rangle^{-2\Delta}. \tag{7.148}$$

Having established the commutative limit, we can claim that our result (7.146) gives a two-parameter deformation of the conformal two-point function at the boundary. Interestingly, this two-point function depends explicitly on both $\langle\hat{t}_1\rangle$ and $\langle\hat{t}_2\rangle$, and not on the difference $\langle\hat{t}_{12}\rangle$. One way to interpret this is to realize this two-point function as a three-point function between two primary operators in a CFT and one non-local defect operator. More precisely, we can compute the following quantity

$$C_\Delta\,\langle\phi(-\tau_1)\phi(\tau_2)\mathcal{D}\rangle, \tag{7.149}$$

where $\tau_j = i\langle\hat{t}_j\rangle$. Field $\phi(\tau)$ is taken to be a primary operator with conformal dimension $\Delta_\phi = \frac{\Delta}{2} - l$, while $\mathcal{D}$ is a non-local defect operator

$$\mathcal{D} = \lambda^{-1}_{\phi\phi\psi}\frac{\Gamma(\Delta-2l)^2}{\Gamma(-2l)^2\Gamma(2\Delta)}\int_{-l\bar{k}}^{\infty} d\tau\,\tau^{-4l-1}\psi(\tau). \tag{7.150}$$

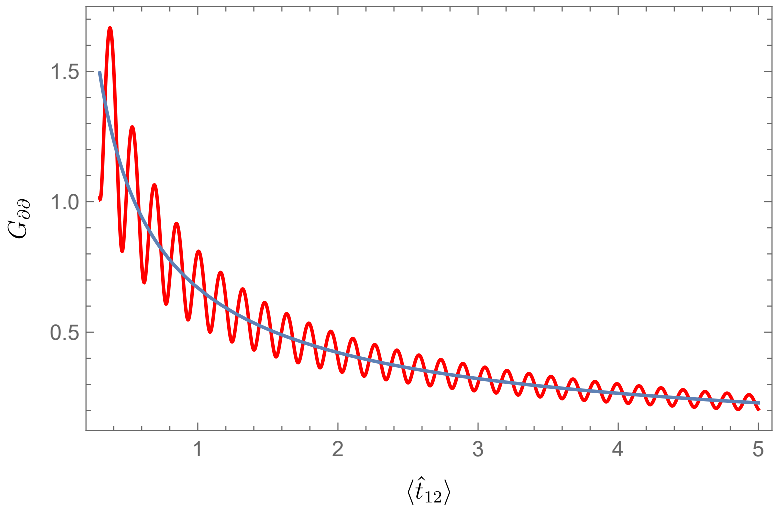

Figure 7.2: The fuzzy two-point function at the boundary (red) vs. the commutative (conformal) two-point function (blue). In this plot, numerical values of constants are: $l = -\infty$, $\lambda\!\!\!^{-} = 1/20$, $\Delta = 1/3$

In this formula, $\psi(\tau)$ is a primary with conformal dimension $\Delta_\psi = 2\Delta_\phi$, and $\lambda_{\phi\phi\psi}$ is the OPE coefficient, appearing in the CFT three-point functions. As explained in Chapter 3, the boundary theory dual to the scalar field on AdS spacetime is the GFF, so we may conjecture that the boundary dual to the scalar field on the fuzzy AdS is again the GFF, but with the insertion of certain defects. Indeed, if we use the fact that

$$\langle \phi(t_1)\phi(t_2)\psi(t)\rangle \;=\frac{\lambda_{\phi\phi\psi}}{|t_1-t_2|^{2\Delta_\phi-\Delta_\psi}|t_1-t|^{\Delta_\psi}|t_2-t|^{\Delta_\psi}}\,, \tag{7.151}$$

and evaluate the integral from (7.150), we obtain that the epression (7.149) mathes the $G_{\partial\partial}$ from (7.146). Of course, at this stage, we are still far away from being able to claim that something similar can be done for any correlation function in the boundary theory, not just the two-point function, but we hope that eventually we should be able to prove this.

Finally, we perform the same computation in the case of the fuzzy AdS_3 bulk, relying on the (7.126) modes. First, we should discuss the normalisation constant c_γ. To begin, we shall remove the factor $\sin 2\varphi_0$ from the (7.126). Previously, we insisted that the normalisation constant should be chosen as the simplest possible option that has the correct commutative limit at the level of modes. However, modes themselves are not observables, and what should have the correct semi-classical limit are precisely objects we should be able to measure. As fuzzy AdS_3 modes are determined by three quantum numbers, compared to the two quantum numbers in the classical AdS_3, it turns out that it is not equivalent to choose c_γ such that the modes have the right semi-classical limit or that the two-point function has the right semi-classical limit. Luckily, the difference between those two normalizations is just a constant independent of the quantum numbers. We choose normalization as

$$c_\gamma = \lambda\!\!\!^{-}\,\frac{2^{2\Delta-8l-\frac{7}{2}}(-l)^{\Delta-4l}\Gamma\left(\frac{1}{2}-2l\right)^{8l}\Gamma(-2l)^{2-8l}\left(f_{\lambda 0}\right)^{1-4l}}{\pi^3\Gamma(\Delta-4l)\sin 2\varphi_0}\;. \tag{7.152}$$

Of course, as before, any other choice of normalization that has the same limiting form in the $l\to-\infty$ limit would suffice. We shall commit to this choice in the following. Note that the

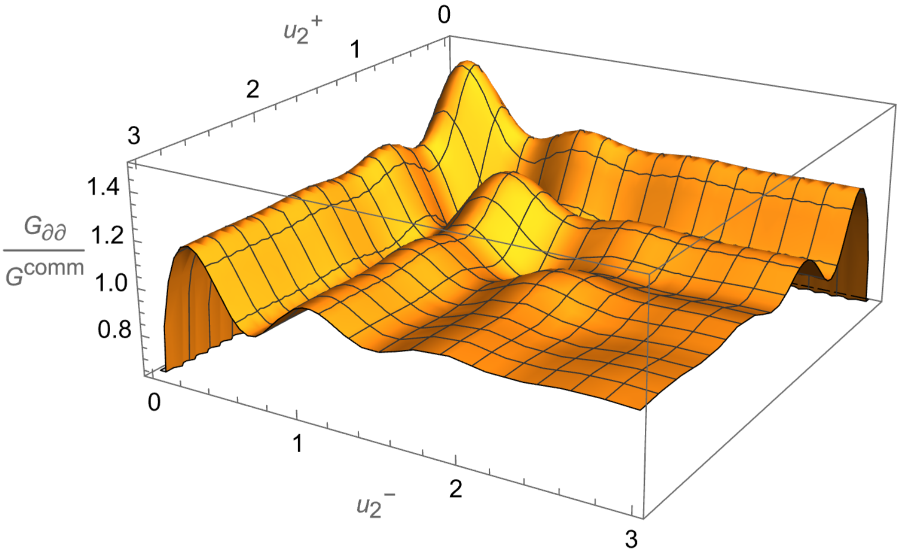

Figure 7.3: Two-point function at the boundary of the fuzzy AdS_3, divided by the commutative two-point function.

boundary limit of modes (7.126) is given by

$$\frac{\langle \phi_\gamma \rangle}{\langle \hat{z} \rangle^\Delta} = \frac{2^{\frac{1}{2}-\Delta}}{\pi \Gamma(\Delta)} \left(\omega^2 - k^2\right)^{\frac{\nu}{2}} \left(1 - \frac{i\omega\langle \hat{t}\rangle}{4l} + \frac{ik\langle \hat{x}\rangle}{4l}\right)^{4l-1-\nu} . \tag{7.153}$$

The two-point function then takes the form of

$$G_{\partial\partial} = \iiint \mathrm{d}\omega \, \mathrm{d}k \, \mathrm{d}\varphi_0$$
$$\frac{2^{1-2\Delta}}{\pi^2 \Gamma(\Delta)^2} \left(\omega^2 - k^2\right)^{\Delta - 1} \left(1 + \frac{i\omega\langle \hat{t}_1\rangle}{4l} - \frac{ik\langle \hat{x}_1\rangle}{4l}\right)^{4l-\Delta} \left(1 - \frac{i\omega\langle \hat{t}_2\rangle}{4l} + \frac{ik\langle \hat{x}_2\rangle}{4l}\right)^{4l-\Delta} .$$

As for the region of integration, we again can show that $|\omega| < \frac{2}{\hbar}$ and $|k| < \frac{2}{\hbar}$. However, $\gamma^2 > 0$ implies $\omega^2 - k^2 \geq 0$. Finally, integration over positive energy modes is enforced by $\omega > 0$ condition, so that the integration region is given by the triangle, determined by the points $(0,0)$, $\left(\frac{2}{\hbar}, -\frac{2}{\hbar}\right)$ and $\left(\frac{2}{\hbar}, \frac{2}{\hbar}\right)$. Finally, the integration region is independent of the quantum number φ_0, so that it is trivial to perform the integration over this quantum number.
In order to perform the computation, we change the variables to the light-cone coordinates

$$k_\pm = \omega \pm k \,, \qquad u^\pm = \frac{1}{2}\left(\langle \hat{t}\rangle \pm \langle \hat{x}\rangle\right) , \tag{7.154}$$

so that the boundary two-point function is given by

$$G_{\partial\partial} = \frac{2^{-1-2\Delta}}{\pi \Gamma(\Delta)^2} \int_0^{4\hbar^{-1}} \mathrm{d}k_+ \left(1 + \frac{ik_+ u_1^-}{4l}\right)^{4l-\Delta} \left(1 - \frac{ik_+ u_2^-}{4l}\right)^{4l-\Delta} \tag{7.155}$$
$$\times \int_0^{4\hbar^{-1}-k_+} \mathrm{d}k_- (k_+ k_-)^{\Delta-1} \left(1 + \frac{ik_- u_1^+}{4l + ik_+ u_1^-}\right)^{4l-\Delta} \left(1 - \frac{ik_- u_2^+}{4l - ik_+ u_2^-}\right)^{4l-\Delta} .$$

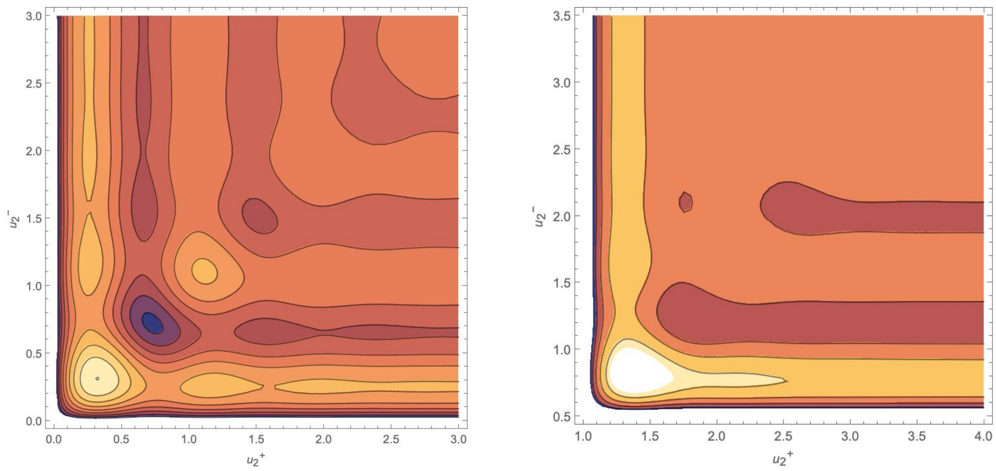

Figure 7.4: Two-point function at the boundary of the fuzzy AdS_3, divided by the commutative two-point function: $(u_1^+, u_1^-) = (0,0)$ vs. $(u_1^+, u_1^-) = (1, 1/2)$.

The integration over k_- is analogous to the two-dimensional case, resulting in the Appell F_1 function

$$
\begin{aligned}
G_{\partial\partial} =& \frac{2^{-1-2\Delta}}{\pi\Gamma(\Delta)^2} \int_0^{4\bar{k}^{-1}} \mathrm{d}k_+ \, k_+^{\Delta-1} \left(1 + \frac{ik_+ u_1^-}{4l}\right)^{4l-\Delta} \left(1 - \frac{ik_+ u_2^-}{4l}\right)^{4l-\Delta} \qquad (7.156)\\
& \times \left(4\bar{k}^{-1} - k_+\right)^{\Delta} F_1\left(\Delta, \Delta-4l, \Delta-4l, \Delta+1; \frac{-i\left(4\bar{k}^{-1}-k_+\right)u_1^+}{4l+ik_+u_1^-}, \frac{i\left(4\bar{k}^{-1}-k_+\right)u_2^+}{4l-ik_+u_2^-}\right) .
\end{aligned}
$$

Unfortunately, we are not able to analytically compute the resulting integral. We can, however, expand the integrand in series over k_+ and then perform the integration. In the leading order, we get

$$
\begin{aligned}
G^{(0)}_{\partial\partial} =& \frac{2^{-1-2\Delta}}{\pi\Gamma(\Delta+1)^2}\left(\frac{4}{\bar{k}}\right)^{2\Delta} F_1\left(\Delta, \Delta-4l, \Delta-4l, \Delta+1; \frac{-iu_1^-}{l\bar{k}}, \frac{iu_2^-}{l\bar{k}}\right) \qquad (7.157)\\
& \times F_1\left(\Delta, \Delta-4l, \Delta-4l, \Delta+1; \frac{-iu_1^+}{l\bar{k}}, \frac{iu_2^+}{l\bar{k}}\right) ,
\end{aligned}
$$

with the higher corrections suppressed in the semi-classical limit by the powers of l^{-1} and $\bar{k}$. Therefore, relying on the results from the fuzzy AdS_2 case, we have

$$
\begin{aligned}
\lim_{\bar{k}\to 0} \lim_{l\to -\infty} G^{(0)}_{\partial\partial} &= \lim_{\bar{k}\to 0} \frac{2^{-1-2\Delta}}{\pi\Gamma(\Delta+1)^2}\left(\frac{4}{\bar{k}}\right)^{2\Delta} {}_1F_1\left(\Delta, \Delta+1; \frac{4iu_{12}^-}{\bar{k}}\right) {}_1F_1\left(\Delta, \Delta+1; \frac{4iu_{12}^+}{\bar{k}}\right)\\
&= \frac{2^{-2\Delta}}{2\pi}\left(-iu_{12}^-\right)^{-\Delta}\left(-iu_{12}^+\right)^{-\Delta} , \qquad (7.158)
\end{aligned}
$$

which matches the conformal two-point function

$$
G^{\mathrm{comm}}_{\partial\partial}(t_1, x_1; t_2, x_2) = \frac{1}{2\pi}\frac{1}{(-t_{12}^2 + x_{12}^2)^{\Delta}} = \frac{1}{2\pi}\frac{1}{(-4u_{12}^- u_{12}^+)^{\Delta}} . \qquad (7.159)
$$

Again, (7.156) provides a two-parameter deformation of the conformal two-point function, now in two spacetime dimensions. Relying on the numerical integration, the ratio of (7.156) and conformal two-point function is plotted in Figure 7.3, with the first point being the origin $u_1^\pm = 0$. Finally, in Figure 7.4 we plot this ratio simultaneously, with the first point at the origin and the second point displaced from the origin. It is evident that in the latter case, the two-point function is not symmetric in $u^\pm$.

Chapter 8

Conclusion

In this thesis, we considered two types of deformed AdS spaces: either deformed by the presence of torsion or deformed by the presence of noncommutativity. Our goal was to perform the holographic analysis of those bulk theories in the spirit of bottom-up AdS/CFT and gain a better understanding of their boundary duals. The majority of work has been done using the gauge-theoretic formulation of certain (Chern-Simons) gravitational theories, and along the way, we gained a better understanding of the first-order formulation of gravity on AdS spacetime. Let us summarize our main findings. First, we analyzed a boundary dual of even-dimensional gravity theory in the first-order formalism and (potentially) propagating torsion: the CTG model. Apart from concrete expressions for the one-point functions of boundary operators (4.52), we were able to obtain a better understanding of boundary terms required to obtain the desired result. As a consistency check, we showed that our findings can be applied in $2D$, where CTG coincides with the BF formulation of JT gravity, and derived correct expressions for the energy and the entropy of $1+1$ D BTZ black hole (4.107). We further investigated the nature of boundary terms in the BF formulation of JT gravity, deriving the profile of the E.o.W. brane directly in the first-order formulation of the theory. Luckily, the profile of the brane is given by a geodesic, as is the case in the standard second-order formulation of JT gravity.

We tried to give a comprehensive discussion on the different approaches to the boundary terms in the Lagrangian formulation of first-order gravity theories with torsion. Unlike in the Hamiltonian formalism, where the first-order formulation was often used in combination with boundary terms, such as to reproduce differentiable generators and finite charges, a systematic treatment of boundary terms in the Lagrangian formulation has not been presented prior to the beginning of this PhD work. One promising candidate for the final word on the boundary terms emerged in late 2022 [31], and we used their reasoning for some of our results. Despite this paper, as there is no well-established consensus on boundary terms, we focused on examples where different approaches yield the same answer, and obtained the E.o.W. brane profile in $3D$ gravity with translational Chern-Simons term. This theory is a particular case of the MB model, with the bulk spacetime necessarily possessing non-zero torsion. By this, we extended the standard AdS/BCFT analysis to the realm of Riemann-Cartan geometry. We discussed why the derived result for the brane profile is expected, confirming our analysis. Further, we discussed the boundary entanglement entropy in this model, where we again obtained consistent results. Having some basic understanding of the boundary dual of this simplified model with torison, we were able to attack a more serious example of five-dimensional CS gravity. As we are often interested in spherically symmetric solutions to the equations of motion, we first performed a dimensional reduction over $\mathbb{S}^3$ to obtain an effective two-dimensional model. A novelty of our approach is that we did not assume vanishing torsion, and even the $\mathbb{S}^3$ part is endowed with nontrivial (axial) torsion, parametrized by the field Φ in (5.30). This field, in turn, has an interpretation of being, when the torsion of the two-dimensional manifold vanishes (which is true for the black hole (2.73)), a coupling between the Riemannian sector of the dimensionally reduced five-dimensional CS gravity with the JT gravity. Using this fact, we extracted the entropy of the black hole with torsion and found that torsion plays a nontrivial role in it. Further discussion was made on the derivation of the entropy, comparing different approaches (some of which are justifiable only in the CS supergravity), all yielding the same answer. We were then able to make a conjecture on the holographic entanglement entropy in a (nonunitary) boundary dual to the five-dimensional CS gravity.

Having in mind our goal to understand the role of torsion in the context of holography, we then engaged ourselves in studying AdS/CMT (Anti de-Sitter/condensed matter theory). We analysed a $U(1)$ bulk gauge field in the probe limit, which introduces a conserved current in the boundary theory, and thus $U(1)$ global symmetry. By introducing a nonminimal coupling between torsion and the gauge field in the bulk (motivated by considerations in the literature on cosmology and similar fields), we were able to compute the conductivity in the boundary theory, related to this conserved current. We established that the torsion (2.81) introduces a finite Drude peak in the optical conductivity in the probe limit. In order to compute the AC optical conductivity, we employed a numerical shooting method. To establish the validity of the obtained results, we compared our numerical analysis with the DC conductivity, which we computed using analytical methods, and with the Riemannian conductivity, which we were also able to analytically compute. We provided examples of real materials that exhibit similar behavior of AC conductivity.

Finally, we tackled the issue of noncommutative gravity. Starting from the noncommutative extension of gauge-theoretic CS gravity in five dimensions, we performed a dimensional reduction to obtain four-dimensional gravity with a negative cosmological constant and discussed how NC effects can modify the affine structure of spacetime by introducing torsion. Furthermore, we showed that this specific spacetime has a nonzero Pontryagin density, responsible for the chiral gravitational anomaly. In order to obtain a NC extension of CS action, we employed the Seiberg-Witten map. Next, we analysed a theory of a scalar and $U(1)$ gauge field on the AdS RN black-hole background and again used the SW mapping to extract the equations governing the dynamics of the scalar field. We showed that the obtained equation can be written as a commutative equation for a scalar field in a modified, effective metric. We then went further with our analysis of NC AdS physics and used a complementary approach, the frame formalism. We considered the fuzzy AdS spaces in two and three dimensions and analysed a scalar field on this NC spacetime. Our goal was to obtain the boundary two-point correlation function using the boundary limit (the extrapolate dictionary of AdS/CFT). In order to do so, we had to introduce the concept of a boundary on a quantum spacetime, which was done through the notion of semiclassical states. Construction of fuzzy AdS_2 and AdS_3 spaces, and of the semiclassical states, depends on two quantum numbers, one labeling the representation of $\mathfrak{sl}(2,\mathbb{R})$ algebra, and the other one measuring the noncommutativity between the coordinates. We showed that the particular limit of those two numbers represents a classical limit, in which the expectation values of field modes in semiclassical states reduce to the commutative ones. By assuming standard rules of quantization in terms of a Fock space, we defined a quantum scalar field on quantum spacetime and computed its two-point function. The boundary two-point functions obtained are analytical in the case of two-dimensional bulk and are expressed as finite integrals in three dimensions, and constitute a two-parameter deformation of the standard CFT two-point functions. Having stated all of these results, we believe that the initial goal of the PhD thesis has been successfully realized.

Even though the results presented here have improved our understanding of holography on deformed spacetimes of the type we described, they also opened many new research directions. First, it is very important that, in the future, we manage to analyse other gravity theories where torsion need not be zero, but which are more nontrivial than CS gravity and CTG. This is also related to the fact that we still lack a systematic treatment of the boundary spin current and its role in applied holography. It would also be beneficial to analyse torsion from a top-down perspective, using string theory. It is a well-known fact that there is a close relation between torsion and the two-form field B in the supergravity action [35]. Therefore, it would be interesting to analyse the NS5-F1 brane system [165] and its near-brane limit to investigate the holographic effects of torsion from the top-down perspective. Furthermore, as mentioned in chapter 3, there is a close relation between AdS/BCFT, entanglement entropy, and the information loss paradox. As the thesis discusses AdS/BCFT and entanglement entropy on Riemann-Cartan manifolds, a natural extension that we plan to do in the future is to explore this interesting topic and the influence of torsion on the behaviour of the Page curve. Finally, we should be able to either prove or disprove the proposal for the computation of the boundary entanglement entropy made in chapter 5.3.

Yet another research direction originating from this work is the analysis of quantum AdS spacetimes as the bulk in some version of AdS/CFT. We have been focused on a NC spacetimes as the bulk, and precisely for this reason, we initiated the study of Riemann-Cartan spacetime (as we saw, there is a close relation between Riemann-Cartan formalism and NC gravity models presented in the thesis). However, NC geometry is by no means the only candidate for the description of quantum spacetimes. Apart from the obvious example of loop quantum gravity [166], where one can define a concrete notion of a quantum spacetime, recently, there has been a breakthrough in the relativistic quantum information community (RQI) on the bottom-up definition of quantum spacetime [84]. The RQI community is particularly interested in the notion of quantum reference frames [167], a generalization of reference-frame transformations to accommodate the phenomenology of quantum superpositions. Motivated by this technique, [84] suggested how to construct a superposed state of two BTZ black holes with different masses and how to analyse the response of the Unruh-de-Witt detector interacting with a scalar field in this quantum background. As we spent quite some time discussing the BTZ black hole, it would be interesting to try to merge our holographic considerations with the considerations from [84]. Actually, even a simpler example is provided by the thermal AdS space. While both thermal AdS and BH spacetime introduce a finite temperature at the boundary (for $D > 3$ they thermodynamically compete for stability, though this is not relevant for our proposal), thermal AdS is much simpler to study as it is (Euclidean) AdS with a periodic Euclidean time. We can study a quantum scalar field on top of the thermal AdS spacetime and obtain the boundary limit of two-point functions. If we use a classical AdS spacetime, the resulting boundary theory is precisely the GFF in the thermal state, and the two-point function is computed using the method of images [95]. We initiated this consideration in a project with Robert Mann. A natural question is then whether we can use our understanding of Fuzzy AdS spacetime to define the fuzzy thermal AdS and compute the deformed boundary thermal two-point function. We hope that we will be able to tackle this serious question and make a contribution to the fascinating topic of thermal CFTs.

To summarize, this thesis offers a good starting point for future holographic considerations of deformed AdS spaces. A key message that should be inferred from the thesis is that the boundary terms in the first-order formulation of gravity are important (and nontrivial for the analysis), that there are cases where torsion does not change the Riemannian results from the Riemannian ones, but also those cases, such as $5D$ CS gravity, where the contribution from torsion can be nontrivial, and finally, that we can indeed find examples of quantum AdS spacetimes where the boundary theory can be analysed. The most important result of this thesis, we believe, is a considerable amount of new research directions that can now be pursued and that we wish to pursue in the future. Whether or not the full scientific potential of the considerations made in this thesis will be reached depends on many factors, some of which are out of our hands. We hope that, sometime soon, the global high-energy community will become more interested in Riemann-Cartan holography and NC AdS bulk spacetimes, as we believe that many important results are yet to be discovered.

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APPENDICES

Appendix A

AdS algebra and representations

In this appendix, we wish to provide an overview of the representation theory for AdS algebras used in the dissertation. Before we begin, we shall make a caveat about the notation used in this work. Unfortunately, as emphasised in the bulk of the thesis, we change the number of spacetime dimensions repeatedly, so we had to use slightly different conventions in different sections. A reader is advised to carefully read the introduction to each section to check if a new piece of notation was introduced. This is particularly important for Chapter 4. However, there are some global features that are present in any calculation not forcing us to repeatedly change the spacetime dimensionality, and we summarize them here. Greek indices $\mu, \nu, \ldots$ are spacetime indices and take values $0, \ldots, D-1$, where D stands for the number of bulk spacetime dimensions. Greek indices $\alpha, \beta, \ldots$ are reserved for boundary spacetime indices. For example, in the Fefferman-Graham gauge, $\mu = (1, \alpha)$. Latin indices $a, b, c, \ldots$ stand for local Lorentz indices in the bulk, while Latin indices $i, j, k, \ldots$ are boundary local Lorentz indices. Through the text, we work with the mostly + metric signature, meaning $\eta_{ab} = (-, +, \ldots, +)$. Levi-Civita tensor is denoted as $\varepsilon_{abc\ldots}$, with convection $\varepsilon_{012\ldots} = 1$. Levi-Civita symbol $\varepsilon_{\mu\nu\rho\ldots}$ follows the same conventions, while Levi-Civita tensor is denoted by $\tilde{\varepsilon}_{\mu\nu\rho\ldots}$. In the Appendix A.1, we take the convention of Chapter 4.

A.1 AdS algebra

The Anti-de Sitter group is $SO(\mathcal{D}-2, 2)$. First, we take $\mathcal{D} = 6$ (which is the dimension of the flat ambient spacetime in which AdS is embedded). The metric for the ambient space is

$$G_{\hat{A}\hat{B}} = (- + + + + -) \,, \tag{A.1}$$

and we denote generators $J_{\hat{A}\hat{B}}$ ($\hat{A}, \hat{B} = 0, 1, 2, 3, 4, 5$); these are separated into J_{AB} and J_{A5} ($A, B = 0, 1, 2, 3, 4$). Their algebra is

$$[J_{\hat{A}\hat{B}}, J_{\hat{C}\hat{D}}] = G_{\hat{A}\hat{D}} J_{\hat{B}\hat{C}} + G_{\hat{B}\hat{C}} J_{\hat{A}\hat{D}} - G_{\hat{A}\hat{C}} J_{\hat{B}\hat{D}} - G_{\hat{B}\hat{D}} J_{\hat{A}\hat{C}} \,, \tag{A.2}$$

or, in components

$$[J_{AB}, J_{CD}] = G_{AD} J_{BC} + G_{BC} J_{AD} - G_{AC} J_{BD} - G_{BD} J_{AC} \,, \tag{A.3}$$

$$[J_{AB}, J_{C5}] = G_{BC} J_{A5} - G_{AC} J_{B5} \,, \tag{A.4}$$

$$[J_{A5}, J_{C5}] = J_{AC} \,. \tag{A.5}$$

A representation of this algebra is provided by 5-dim gamma matrices Γ_A satisfying Cliford algebra $\{\Gamma_A, \Gamma_B\} = 2G_{AB}\mathbb{I}_{4\times 4}$ with $G_{AB} = (- + + + +)$. In terms of these gamma matrices, we have

$$J_{AB} = \frac{1}{2}\Gamma_{AB} = \frac{1}{4}[\Gamma_A, \Gamma_B]\,, \tag{A.6}$$

$$J_{A5} = \frac{1}{2}\Gamma_A\,. \tag{A.7}$$

Finally, we can define 5-dim gamma matrices in terms of ordinary 4-dim gamma matrices as $\Gamma_A = (-i\gamma_a, \gamma_5)$ with $a = 0, 1, 2, 3$, satisfying $\{\gamma_a, \gamma_b\} = -2G_{ab}\mathbb{I}_{4\times 4}$ with $G_{ab} = (- + ++)$. In other words, for 4-dim gamma matrices we use the standard, QFT conventions; $\{\gamma_a, \gamma_b\} = 2\eta_{ab}\mathbb{I}_{4\times 4}$ with $\eta_{ab} = (+ - --)$. Also, $\Gamma_4 = \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$ and $\gamma_5^2 = \mathbb{I}_{4\times 4}$. Furthermore, relation $\Gamma_{AB} = \frac{1}{2}[\Gamma_A, \Gamma_B]$ entails $\Gamma_{ab} = \frac{1}{2}[\Gamma_a, \Gamma_b] = -\frac{1}{2}[\gamma_a, \gamma_b] = i\sigma_{ab}$ and $\Gamma_{a4} = \frac{1}{2}[\Gamma_a, \Gamma_4] = -\frac{i}{2}[\gamma_a, \gamma_5] = -i\gamma_a\gamma_5$. In terms of the four-dimensional gamma matrices, the generators are given by

$$J_{ab} = \frac{1}{2}\Gamma_{ab} = \frac{i}{2}\sigma_{ab} \tag{A.8}$$

$$J_{a4} = \frac{1}{2}\Gamma_{a4} = -\frac{i}{2}\gamma_a\gamma_5\,, \tag{A.9}$$

$$J_{a5} = \frac{1}{2}\Gamma_a = -i\frac{1}{2}\gamma_a\,. \tag{A.10}$$

$$J_{45} = \frac{1}{2}\Gamma_4 = \frac{1}{2}\gamma_5\,. \tag{A.11}$$

We now state the Hermitian properties of the generators in this representation:

$$J_{ab}^\dagger = -\gamma_0 J_{ab}\gamma_0 = \Gamma_0 J_{ab}\Gamma_0\,, \tag{A.12}$$

$$J_{a4}^\dagger = -\gamma_0 J_{a4}\gamma_0 = \Gamma_0 J_{a4}\Gamma_0\,, \tag{A.13}$$

$$J_{a5}^\dagger = \frac{i}{2}\gamma_0\gamma_a\gamma_0 - \gamma_0 J_{a5}\gamma_0 = \Gamma_0 J_{a5}\Gamma_0\,, \tag{A.14}$$

$$J_{45}^\dagger = \frac{1}{2}\gamma_5 = -\gamma_0 J_{45}\gamma_0 = \Gamma_0 J_{a5}\Gamma_0\,, \tag{A.15}$$

that is

$$J_{AB}^\dagger = \Gamma_0 J_{AB}\Gamma_0\,, \tag{A.16}$$

$$J_{A5}^\dagger = \Gamma_0 J_{A5}\Gamma_0 = \Gamma_0 J_{A5}\Gamma_0\,. \tag{A.17}$$

We define $\varepsilon_{01234} = +1$ and $\varepsilon^{01234} = -1$ (and rise and lower indices with G_{AB}). From the general relation $\varepsilon^{A_1A_2\ldots A_{n-1}A}\varepsilon_{A_1A_2\ldots A_{n-1}B} = (-1)^s(n-1)!\delta^A_B$, where s is the number of minuses in the metric (for G_{AB}, $s = 1$), follows

$$\varepsilon^{ABCDE}\varepsilon_{ABCMN} = -3!(\delta^D_M\delta^E_N - \delta^D_N\delta^E_M)\,, \tag{A.18}$$

$$\varepsilon^{ABCDE}\varepsilon_{ABCDF} = -4!\delta^E_F\,, \tag{A.19}$$

$$\varepsilon^{ABCDE}\varepsilon_{ABCDE} = -5!\,, \tag{A.20}$$

$$\begin{aligned} \mathrm{d}x^{\hat\mu}\wedge\mathrm{d}x^{\hat\nu}\wedge\mathrm{d}x^{\hat\rho}\wedge\mathrm{d}x^{\hat\sigma}\wedge\mathrm{d}x^{\hat\tau} &= -\varepsilon^{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau}\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge\mathrm{d}x^4 = -\varepsilon^{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau}\mathrm{d}x^5 \\ &= -\frac{1}{5!}\varepsilon^{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau}\varepsilon_{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau}\mathrm{d}x^{\hat\mu}\wedge\mathrm{d}x^{\hat\nu}\wedge\mathrm{d}x^{\hat\rho}\wedge\mathrm{d}x^{\hat\sigma}\wedge\mathrm{d}x^{\hat\tau}\,. \end{aligned} \tag{A.21}$$

Finally, we need the trace identities for Γ-matrices:

$$\begin{aligned}
\mathrm{Tr}(\Gamma_A) &= 0\,, &\text{(A.22)}\\
\mathrm{Tr}(\Gamma_A\Gamma_B) &= 4G_{AB}\,, &\text{(A.23)}\\
\mathrm{Tr}(\Gamma_A\Gamma_B\Gamma_C) &= 0\,, &\text{(A.24)}\\
\mathrm{Tr}(\Gamma_A\Gamma_B\Gamma_C\Gamma_D) &= 4(G_{AB}G_{CD} - G_{AC}G_{BD} + G_{BC}G_{AD})\,, &\text{(A.25)}\\
\mathrm{Tr}(\Gamma_A\Gamma_B\Gamma_C\Gamma_D\Gamma_E) &= 4i\varepsilon_{ABCDE}\,. &\text{(A.26)}
\end{aligned}$$

This implies the trace identities for generators:

$$\begin{aligned}
\mathrm{Tr}(J_{AB}J_{CD}J_{E5}) &= \frac{i}{2}\varepsilon_{ABCDE}\,, &\text{(A.27)}\\
\mathrm{Tr}(J_{AB}J_{C5}J_{D5}) &= \frac{1}{2}(G_{AB}G_{CD} - G_{AC}G_{BD} + G_{BC}G_{AD})\,, &\text{(A.28)}\\
\mathrm{Tr}(J_{A5}J_{C5}J_{D5}) &= 0\,. &\text{(A.29)}
\end{aligned}$$

A.2 Discrete series representation of $\mathfrak{sl}(2,\mathbb{R})$ algebra

Here, we present certain mathematical details used in the thesis on unitary irreducible representations of $SO(2,1) \approx SL(2,\mathbb{R})$. Lie algebra for this groups has three generators, labeled $\{H, E_\pm\}$, satisfying the algebra relations

$$[H, E_+] = E_+\,, \quad [H, E_-] = -E_-\,, \quad [E_+, E_-] = 2H\,. \tag{A.30}$$

If one makes the following linear combination of them

$$\tilde{H} = \frac{i}{2}(E_+ - E_-)\,, \qquad \tilde{E}_+ = \frac{1}{2}(E_+ + E_- + 2iH)\,, \qquad \tilde{E}_- = \frac{1}{2}(E_+ + E_- - 2iH)\,, \tag{A.31}$$

we see that the form of the commutation relations does not change. Note that $\tilde{H}$ is the generator of the maximal compact subgroup of $SL(2,\mathbb{R})$. This implies that in any unitary irreducible representation, this operator will have a discrete spectrum. This group has three series of infinite-dimensional unitary irreducible representations: the principal continuous series, the complementary series, and the discrete series. We shall discuss only the discrete series representations, as they are relevant for this thesis. We denote by T_l^- and T_l^+ the two families of discrete series representations [132], with l being a half-integer or an integer. In the case of T_l^-, we have $l \leq -1$, while for T_l^+ we have $l \geq 1$. We denote the vector space on which the T_l^- acts as $\mathcal{H}_l$. Importantly, T_l^- and T_l^+ are related as

$$\left(T_l^-\right)^* = T_{-l}^+\,, \tag{A.32}$$

with the star denoting the dual (contragredient) representation. Alternatively, we can relate those two representations using the *Bargmann automorphism*, B

$$\left(T_l^-\right)^B = T_{-l}^+ \tag{A.33}$$

This is an outer automorphism of $SL(2,\mathbb{R})$ and on generators its action is given by

$$B(H) = H\,, \qquad B(E_+) = -E_+\,, \qquad B(E_-) = -E_-\,. \tag{A.34}$$

The discrete series representations have the property that they are the highest/lowest vector representations, a property not shared among other types of unitary irreducible representations of $\mathfrak{sl}(2,\mathbb{R})$. T_l^- has the lowest-weight vector Ψ_0, such that

$$\tilde{E}_-\Psi_0 = 0\,, \qquad \tilde{H}\Psi_0 = -l\Psi_0\,, \tag{A.35}$$

with other vectors in the representation taking the form of linear combinations of elements $\Psi_n = \tilde{E}_+^n\Psi_0$.

One way to realize the representation T_l^- is on the space of square-integrable functions $f(\xi)$, where $\xi > 0$, with the inner product

$$(f_1, f_2) = 2^{2l+1}\pi \int_0^\infty \mathrm{d}\xi\, \xi^{2l+1} f_1(\xi)^* f_2(\xi)\ . \tag{A.36}$$

A concrete realization of the algebra generators is then given as

$$H = \xi\partial_\xi + l + 1\,, \qquad E_+ = -i\xi\,, \qquad E_- = -i\left(\xi\partial_\xi^2 + 2(l+1)\partial_\xi\right)\ . \tag{A.37}$$

As standard in quantum mechanics, we define $f(\xi) = \langle\xi|f\rangle$. We then have

$$\langle\xi|\xi'\rangle = (\pi\, 2^{2l+1}\xi^{2l+1})^{-1}\,\delta(\xi - \xi')\,, \tag{A.38}$$

with $\delta(\xi - \xi')$ being the Dirac δ-function.

Finally, we can use the isomorphism

$$\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})\ . \tag{A.39}$$

to address the $\mathfrak{so}(2,2)$ representations. They are given as the tensor product $\pi_1 \otimes \pi_2$ of $\mathfrak{sl}(2,\mathbb{R})$ representations π_i. In this thesis, we shall make the distinction between two copies of $\mathfrak{sl}(2,\mathbb{R})$ by putting the bar on all the quantities related to the second copy. One important realization of $T_l^- \otimes \bar{T}_l^-$ is given on the space of two-variable functions $f(\xi,\bar{\xi})$ (the scalar product is inferred from (A.36)). This representation has the lowest-weight vector of the form

$$\Psi_0 \otimes \bar{\Psi}_0 = N^2(\xi\bar{\xi})^{-2l-1}e^{-\xi-\bar{\xi}}\ . \tag{A.40}$$

Biografija

Dušan Đorđević rođen je 1998. godine u Beogradu. Osnovnu školu „Braća Vilotijević" završio je 2013. godine u Kraljevu, dok je „Matematičku gimnaziju" u Beogradu završio 2017. godine, osvojivši te godine zlatnu medalju na Međunarodnoj fizičkoj olimpijadi (*IPhO*) u Indoneziji. Kao student generacije, 2020. godine završava osnovne studije fizike na Fizičkom fakultetu Univerziteta u Beogradu, na smeru teorijska i eksperimentalna fizika, sa prosečnom ocenom 10.00. Naredne, 2021. godine završava master studije iz matematičke i teorijske fizike na Univerzitetu u Oksfordu, sa najvišom mogućom ocenom (*distinctions*) i nagradom profesorskog kolegijuma. Iste godine završava i master studije iz teorijske i eksperimentalne fizike na Fizičkom fakultetu Univerziteta u Beogradu, sa prosečnom ocenom 10.00, odbranivši master rad na temu *"Noncommutative Five-Dimensional Chern-Simons Gravity"*. Oktobra 2021. godine upisao je doktorske studije na Fizičkom fakultetu Univerziteta u Beogradu. Od maja 2022. godine je zaposlen na ovom fakultetu, a u zvanje istraživač-saradnik izabran je krajem novembra 2024. godine. Dobitnik je godišnje nagrade za naučni rad mladih istraživača Fizičkog fakulteta 2024. godine.

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