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Casimir operators and group projectors

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Abstract. The technique for determination of the symmetry adapted bases for finite-dimensional representations of the Lie groups is developed, in analogy to the finite group theory. The method uses the corresponding Lie algebra, and relates the group projectors to the Casimir operators. As an application to the semisimple algebras, the general formula for generating function of the values of the Casimir operators is established.

1. Introduction

The problem of finding the standard, or the symmetry adapted basis, frequently appears in different physical theories. As for the finite groups, it is solved by the group projector technique. The same method, involving the summation over the group, is possible, although rarely applied, for the compact Lie groups, while for the other groups such procedure cannot be extended. On the other hand, the recently developed modification of the group projector technique [1, 2] avoids the summation over group, using for each irreducible component $D^{(\mu)}(G)$ of the representation $D(G)$ the single group projector $G(D \otimes D^{(\mu)*})$, which can be calculated with the help of the generators of the group only. Therefore, it is natural to attempt to generalize this procedure to Lie groups.

Such generalization is the main aim of this paper. Due to their relevance, and in order to preserve the clarity of the idea, the scope of this study is restricted to the finite-dimensional decomposable representations, which enables one to avoid the cohomology theory [3]. Within this framework, the semisimple groups are the most interesting examples, the more so because of the Casimir operators theory, which for such groups solves some neighbouring problems. After introducing basic concepts and notation (section 2), the algorithm is formulated through the single Casimir operator, and the method of generating the series of Casimir operators is explained in section 3. Finally, several examples are studied to point out some common characteristics of the approach.

2. Group projectors

Let $D_1(G)$ and $D_2(G)$ be representations of the group G, acting in the $|D_1|$- and $|D_2|$-dimensional spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The dual space $\mathcal{H}_1'$ of $\mathcal{H}_1$ carries the representation $D_1'(G)$, dual (or contragredient) to $D_1(G)$: $D_1'(g) = D_1^{T^{-1}}(g)$ (for unitary $D(G)$ it is equal to the complex conjugated representation). This defines the product

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representation in the space $\mathcal{H}_2 \otimes \mathcal{H}'_1$. The subspace $\mathcal{F}$ of the fixed points for this action is the invariant subspace in which the $D_2(G) \otimes D'_1(G)$ is reduced to the (multiple) identity representation. If Q is the isomorphism of the space $\mathcal{H}_2 \otimes \mathcal{H}'_1$ onto the space $\mathrm{Hom}(\mathcal{H}_1, \mathcal{H}_2)$ (of the linear operators mapping $\mathcal{H}_1$ into $\mathcal{H}_2$), the group action in the latter is naturally imposed by

$$gT = Q(D_2(g) \otimes D'_1(g))Q^{-1} \qquad T = D_2(g)TD_1(g^{-1}) \qquad T \in \mathrm{Hom}(\mathcal{H}_1, \mathcal{H}_2). \tag{1}$$

Obviously, Q maps $\mathcal{F}$ onto the space of the intertwining operators, $\mathrm{Hom}_G(\mathcal{H}_1, \mathcal{H}_2)$: $Q\mathcal{F} = \mathrm{Hom}_G(\mathcal{H}_1, \mathcal{H}_2)$. Hence, the dimension of $\mathcal{F}$ is equal to the intertwining number [4].

Let the vectors of the basis $\{|i;1\rangle \mid i = 1, \ldots, |D_1|\}$ in $\mathcal{H}_1$ transform according to $D_1(g)|i;1\rangle = \sum_{j=1}^{|D_1|} D_{1ji}(g)|j;1\rangle$. Recall that the dual basis $\{\langle i;1| \mid i = 1, \ldots, |D_1|\}$ in $\mathcal{H}'_1$ is given by $\langle j;1|i;1\rangle = \delta_{ij},\ i, j = 1, \ldots, |D_1|$. Then for each $|x\rangle$ in $\mathcal{H}_2 \otimes \mathcal{H}'_1$, the vectors $|i;2,x\rangle, i = 1, \ldots, |D_1|$, from $\mathcal{H}_2$ are uniquely defined by

$$|x\rangle = \sum_{i=1}^{|D_1|} |i;2,x\rangle \otimes \langle i;1|. \tag{2}$$

In particular, for $|x\rangle \in \mathcal{F}$, the condition $D_2(g) \otimes D'_1(g)|x\rangle = |x\rangle$, in the form

$$\sum_{i=1}^{|D_1|}(D_2(g)|i;2,x\rangle) \otimes \left(\sum_{j=1}^{|D_1|} D'_{1ji}(g)\langle j;1|\right) = \sum_{j=1}^{|D_1|} |j;2,x\rangle \otimes \langle j;1|$$

gives $|j;2,x\rangle = D_2(g)\sum_{i=1}^{|D_1|} D'_{1ji}(g)|i;2,x\rangle$. Multiplying both sides with $D_2(g^{-1})$, and substituting g by g^{-1}, it follows that

$$D_2(g)|i;2,x\rangle = \sum_{j=1}^{|D_1|} D_{1ji}(g)|j;2,x\rangle. \tag{3}$$

From the operator point of view, expression (2) means that the vector $|x\rangle \in \mathcal{F}$ is by the isomorphism Q mapped into the operator $Q(|x\rangle) \in \mathrm{Hom}_G(\mathcal{H}_1, \mathcal{H}_2)$, having, in the Dirac notation, the same form as $|x\rangle$. Then (3) shows that $|i;1\rangle$ and its 'twin' vector $|i;2,x\rangle = Q(|x\rangle)|i;1\rangle$ have the same transformation properties.

In what follows, $D_1(G)$ is the irreducible representation, $D^{(\mu)}(G)$, in the space $\mathcal{H}_1 = \mathcal{H}_{(\mu)}$. This ensures independence of the vectors $\{|j;2,x\rangle \mid j = 1, \ldots, |D_1|\}$, since they transform according to the irreducible representation. Also, $D_2(G)$ and $\mathcal{H}_2$ are denoted by $D(G)$ and $\mathcal{H}$. When the matrices of $D^{(\mu)}(G)$ are given, the standard basis $\{|\mu m\rangle | m = 1, \ldots, |\mu|\}$ in $\mathcal{H}_{(\mu)}$ is defined by the group action

$$D^{(\mu)}(g)|\mu m\rangle = \sum_{m'=1}^{|\mu|} D^{(\mu)}_{m'm}(g)|\mu m'\rangle. \tag{4}$$

Schur's lemma easily shows that the vectors of this basis are uniquely determined, up to a common constant.

Let the basis in the subspace $\mathcal{F}^{(\mu)}$ of the fixed points of $D(G) \otimes D^{(\mu)'}(G)$ in $\mathcal{H} \otimes \mathcal{H}'_{(\mu)}$ be

$$\{|\mu t_\mu\rangle | t_\mu = 1, \ldots, |\mathcal{F}^{(\mu)}|\}. \tag{5}$$

According to (2), each of these vectors uniquely determines the vectors

$$\{|\mu t_\mu m\rangle | m = 1, \ldots, |\mu|\} \tag{6}$$

such that

$$|\mu t_\mu\rangle = \sum_{m=1}^{|\mu|} |\mu t_\mu m\rangle \otimes \langle \mu' m|. \tag{7}$$

In the case that the scalar products are suitably defined in all the spaces, and the orthonormal bases are dealt with, the right-hand side of (7) should be multiplied by the constant $1/\sqrt{|\mu|}$. Transforming under $D(G)$ like $|\mu m\rangle$ in (4), the vectors (6) are the standard subbasis in $\mathcal{H}$, and span the irreducible invariant subspace $\mathcal{H}^{(\mu t_\mu)}$. The direct sum $\mathcal{H}^{(\mu)} = \oplus_{t_\mu=1}^{|\mathcal{F}^{(\mu)}|} \mathcal{H}^{(\mu t_\mu)}$ is the maximal invariant subspace in $\mathcal{H}$ carrying the multiple ($|\mathcal{F}^{(\mu)}|$ times) of the representation $D^{(\mu)}(G)$. Finally, the total standard basis in $\mathcal{H}$ is

$$\{|\mu t_\mu m\rangle | \mu; t_\mu = 1, \ldots, |\mathcal{F}^{(\mu)}|; m = 1, \ldots, |\mu|\}. \tag{8}$$

To summarize, the procedure to obtain the standard basis consists of two steps. First, the basis $\{|\mu t_\mu\rangle\}$ of the subspace $\mathcal{F}^{(\mu)}$ for each irreducible component is to be found. Afterward, the standard subbasis in each $\mathcal{H}^{(\mu)}$ is derived: for each vector $|\mu t_\mu\rangle$, the operator $Q(|\mu t_\mu\rangle)$ from $\mathcal{H}_{(\mu)}$ to $\mathcal{H}^{(\mu)}$, maps the standard basis $|\mu m\rangle$ into

$$|\mu t_\mu m\rangle = Q(|\mu t_\mu\rangle)|\mu m\rangle. \tag{9}$$

Within Dirac's notation (7), this relation becomes the partial scalar product (only in $\mathcal{H}_{(\mu)}$) of $|\mu t_\mu\rangle$ with $|\mu m\rangle$, which can be found straightforwardly. Thus, the problem of deriving the standard basis (8) effectively reduces to finding the basis (5) in $\mathcal{H} \otimes \mathcal{H}'_{(\mu)}$.

To shorten the notation, let $\Gamma(G) = D(G) \otimes D^{(\mu)'}(G)$ and $\mathcal{H}_\Gamma = \mathcal{H} \otimes \mathcal{H}'_{(\mu)}$. The projector onto the subspace $\mathcal{F}^{(\mu)}$ of the fixed points is $G(\Gamma) = \sum_{t_\mu} |\mu t_\mu\rangle\langle \mu t_\mu|$. The vectors of $\mathcal{F}^{(\mu)}$ are the eigenvectors for eigenvalue 1 of all the operators of the representation $\Gamma(G)$. Then the projector $G^{(\mu)}(D)$ onto the space $\mathcal{H}^{(\mu)}$ is related to the projector $G(\Gamma)$ through the partial trace over the space $\mathcal{H}^{(\mu)'}$:

$$G^{(\mu)}(D) = |\mu| \operatorname{Tr}_\mu G(D \otimes D^{(\mu)'}). \tag{10}$$

This relation can be viewed as an operator analogy of (9).

As for the finite groups, since $\mathcal{F}^{(\mu)}$ is the subspace of the (multiple) identity representation of G, we have the usual expression

$$G(\Gamma) = \frac{1}{|G|} \sum_{g \in G} \Gamma(g).$$

Also, (10) becomes the familiar expression (with the characters χ)

$$G^{(\mu)}(D) = \frac{|\mu|}{|G|} \sum_{g \in G} \xi^{(\mu)*}(g) D(g).$$

Nevertheless, the elements of the group are monomials over the generators, and the common fixed points of the generators are automatically the fixed points for the whole group. This has been used to show, [2], that for unitary representations, the subspace $\mathcal{F}^{(\mu)}$ is the kernel $\mathcal{N}(K)$ of the suitably defined operator, determined only by the generators $\{g_1, \ldots, g_\gamma\}$ of G: if H_i are the hermitean operators such that $\Gamma(g_i) = \mathrm{e}^{\mathrm{i}H(g_i)}$, then

$$\mathcal{F}^{(\mu)} = \mathcal{N}(K) \qquad K = \sum_{i=1}^{\gamma} H^2(g_i). \tag{11}$$

Therefore, the group projector takes the suitable form $G(\Gamma) = \lim_{t \to -\infty} \mathrm{e}^{(t/2)K}$. This relation will be generalized to the Lie groups, when the elements of the Lie algebra take the role of the generators.

3. Casimir operators

It is known that the derivatives (in the identity of the Lie group G) of the representation $\Gamma(G)$ are the representation, $\Gamma(L)$, of the corresponding Lie algebra L. At this level, $\mathcal{F}^{(\mu)}$ is characterized as the maximal subspace annihilated by $\Gamma(L)$, since each vector $|x\rangle \in \mathcal{H}_\Gamma$ is invariant under $\Gamma(G)$ if and only if it is annulated by the operators $\Gamma(L)$. If $\{l_1, \ldots, l_{|L|}\}$ is a basis of L, then the annihilator of the $\Gamma(L)$ is equal to the intersection of the kernels $\mathcal{N}(\Gamma(l_i))$. Instead of the calculation of all these subspaces and their intersection, any auxiliary scalar product can be introduced in order to define the adjoined operators $\Gamma^\dagger(l_i)$. Then, analogously to (11)

$$\mathcal{F}^{(\mu)} = \mathcal{N}(K), \quad K(\Gamma) = \sum_{i=1}^{|L|} \Gamma^\dagger(l_i)\Gamma(l_i). \tag{12}$$

Being the sum of the positive operators, $K(\Gamma)$ is positive itself; its kernel is the intersection of the kernels of the addends, while $\mathcal{N}(\Gamma^\dagger(l_i)\Gamma(l_i)) = \mathcal{N}(\Gamma(l_i))$. The group projector can be defined again as $G(\Gamma) = \lim_{t\to-\infty} G(\Gamma, t)$, with $G(\Gamma, t) = \mathrm{e}^{(t/2)K(\Gamma)}$. In this context the operator $K(\Gamma)$ serves instead of the group projector: $\mathcal{F}^{(\mu)}$ and the standard basis are derived with the help of $K(\Gamma)$, as well as the group projector itself. Since $\Gamma(l) = D(l) \otimes I_{\mu'} + I_D \otimes D^{(\mu)'}(l)$, with $D^{(\mu)'}(l) = -D^{(\mu)^T}$, the operator $K(\Gamma)$ becomes

$$K(\Gamma) = K(D) \otimes I_{\mu'} + I_D \otimes K(D^{(\mu)'}) + \sum_{i=1}^{|L|} (D^\dagger(l_i) \otimes D^{(\mu)'}(l_i) + D(l_i) \otimes D^{(\mu)'^\dagger}(l_i)). \tag{13}$$

It may seem strange that the operator $K(\Gamma)$ depends on the scalar product which is arbitrarily defined. However, although $K(\Gamma)$ depends on this product, its kernel, which is the only relevant notion for the proposed algorithm, does not: $\mathcal{F}^{(\mu)}$ is defined *a priori*, through the group action $\Gamma(G)$. This observation resembles the fact that in the general case there is no natural scalar product, from the group theoretical point of view. In the special cases, some underlying scalar product, usually defined by the physical problem, may be applied. The example of the finite and compact groups can be reconsidered from this point of view, understanding the expression (11) as the special case of (12) for unitary $\Gamma(G)$.

The method can be further developed for the semisimple groups. In the complexified algebra we have the Cartan–Weyl's basis $\{l_1, \ldots, l_{|L|}\} = \{h_i, e_\alpha | i = 1, \ldots, r;\ \alpha \in \boldsymbol{A}\}$ (r is the rank of L, and $\boldsymbol{A}$ is the set of $|L| - r$ roots), such that, for the appropriate scalar product in $\mathcal{H}_\Gamma$, $\Gamma(h_i)$ are hermitean, while $\Gamma^\dagger(e_\alpha) = \Gamma(e_{-\alpha})$. Thus (13) becomes the Casimir operator [4, 5]

$$K(\Gamma) = \sum_{i=1}^{r} \Gamma^2(h_i) + \sum_{\alpha\in\boldsymbol{A}} \Gamma(e_\alpha)\Gamma(e_{-\alpha}) \tag{14}$$

commuting with all of the operators $\Gamma(L)$ and $\Gamma(G)$. Obviously, its kernel is the subspace of the irreducible representation with weight 0, i.e. the identity representation of G. Hence, in the proposed technique of finding the standard basis for the representation $\Gamma(G)$, the Casimir operator $K(D \otimes D^{(\mu)'})$ is found for each irreducible component, and the orthonormal basis $|\mu t_\mu\rangle$ in its kernel. Then the standard basis is easily obtained according to (9). This seems to be simpler in comparison to the usual procedure, where the eigen problem for the set of r independent Casimir operators is used to determine only the decomposition of $\mathcal{H}$ onto the multiple irreducible subspaces $\mathcal{H}^{(\mu)}$, while the standard bases within them are looked for independently.

Since $\mathcal{H}^{(\mu)}$ is determined by the projector (10), the derivation of this projector, through the relation between $K(\Gamma)$ and $G(\Gamma)$ and (10), is equivalent to the solution of the common eigen problem of the set of independent Casimir operators for the eigenvalues corresponding to $D^{(\mu)}(G)$. Indeed, equation (14) gives

$$K(\Gamma) = K(D) \otimes I_{\mu'} + I_D \otimes K(D^{(\mu)'}) + 2\sum_{i=1}^{|L|} D(l_i) \otimes D^{(\mu)'\dagger}(l_i). \tag{15}$$

The last term acts on each $|f\rangle \in \mathcal{F}^{(\mu)}$ as $-2I \otimes K(D^{(\mu)'})$: the condition $\Gamma(l_i)|f\rangle = 0$ is used in the form $D(l_i) \otimes I_{\mu'}|f\rangle = -I_D \otimes D^{(\mu)'}(l_i)|f\rangle$. Furthermore, since $I_D \otimes K(D^{(\mu)'}) = k_\mu I$, the equation $K(D) \otimes I_{\mu'}|f\rangle = I_D \otimes K(D^{(\mu)'})|f\rangle$ is automatically fulfilled. This shows that $\mathcal{F}^{(\mu)} < \mathcal{N}(K(D) \otimes I_{\mu'} - k_\mu I)$, implying that $\mathcal{H}^{(\mu)}$ is the subspace of the eigenspace of $K(D)$ for the eigenvalue k_μ, i.e. that each vector $|x\rangle$ from $\mathcal{H}^{(\mu)}$ satisfies

$$K(D)|x\rangle = k_\mu |x\rangle. \tag{16}$$

According to (10), with $G(\Gamma, t) = \mathrm{e}^{(t/2)K(\Gamma)}$, $G^{(\mu)}(D)$ is the limit ($t \to -\infty$) of the operator function

$$\begin{aligned} G^{(\mu)}(D, t) &= |\mu| \operatorname{Tr}_\mu G(D \otimes D^{(\mu)'}, t) \\ &= |\mu| \exp[(t/2)(K(D) + k_\mu I_D)] \operatorname{Tr}_\mu \exp\left[t \sum_{i=1}^{|L|} D(l_i) \otimes D^{(\mu)'\dagger}(l_i)\right]. \end{aligned} \tag{17}$$

Although $G^{(\mu)}(D, t)$ is not a projector for finite t, the vectors $|\mu t_\mu m\rangle$ are its fixed points. The first factor $\exp[(t/2)(K(D) + k_\mu I_D)]$ acts as e^{tk_μ} in $\mathcal{H}^{(\mu)}$, and this subspace must be the eigenspace for the whole series of Casimir operators

$$C_s^{(\mu)}(D) = \sum_{i_1,\ldots,i_s} D(l_{i_1}) \ldots D(l_{i_s}) \operatorname{Tr} D^{(\mu)'\dagger}(l_{i_1}) \ldots D^{(\mu)'\dagger}(l_{i_s}) \tag{18}$$

which obviously appear in the expansion of the last factor:

$$C^{(\mu)}(D, t) = \operatorname{Tr}_\mu \exp\left[t \sum_{i=1}^{|L|} D(l_i) \otimes D^{(\mu)'\dagger}(l_i)\right] = \sum_{s=0}^{\infty} \frac{t^s}{s!} C_s^{(\mu)}(D). \tag{19}$$

Thus, the function (19) is the generating function for the Casimir operators. Also, the function

$$B^{(\mu)}(D, t) = \operatorname{Tr}_\mu \left(1 - t \sum_{i=1}^{|L|} D(l_i) \otimes D^{(\mu)'\dagger}(l_i)\right)^{-1} = \sum_{s=0}^{\infty} t^s C_s^{(\mu)}(D) \tag{20}$$

can be used to obtain a compact form for a number of related expressions [4, ch 9].

4. Examples

In the usual example of the Lie algebra su(2), the maximal weight $\mu = M = 0, \frac{1}{2}, 1, \ldots$ characterizes the $|\mu| = (2M + 1)$ dimensional irreducible representations, with the known matrix form [6]. The matrices of the irreducible representations are chosen such that the Cartan–Weyl's basis $\{H, E_+, E_-\}$ (of the complexified algebra $\mathrm{sl}(2, \mathbb{C})$) is represented by

$$D^{(M)}_{m'm}(H) = m\delta_{m'm} \qquad D^{(M)}_{m'm}(E_\pm) = \sqrt{\frac{(M \mp m)(M \pm m + 1)}{2}}\,\delta_{m',m\pm 1}$$

while $K(M) = M(M+1)I_M$. Relation (16) immediately gives the condition $K(D)|Mt_M m\rangle = M(M+1)|Mt_M m\rangle$. The rank of the algebra is 1, and this equation completely determines the space $\mathcal{H}^{(M)}$.

The next example is the Lorentz algebra, so(1, 3). The irreducible representations of the complexified algebra $\mathrm{so}(1,3)_{\mathbb{C}} = \mathrm{sl}(2, \mathbb{C}) \oplus \mathrm{sl}(2, \mathbb{C})$ are classified according to the pairs

$\mu = (M_1, M_2)$, where M_i is the maximal weight for the corresponding $\mathrm{sl}(2, \mathbb{C})$ ideal. The sum (13) split into sums over ideals: $K(\Gamma) = K_1(\Gamma) + K_2(\Gamma)$. Now, equation (16) for the irreducible component $D^{(M_1,M_2)}(G)$ reads

$$(K_1(D) + K_2(D))|x\rangle = (M_1(M_1 + 1) + M_2(M_2 + 1))|x\rangle.$$

This equation does not determine the subspace $\mathcal{H}^{(M_1,M_2)}$, since it does not distinguish between the representations with the same $M_1(M_1 + 1) + M_2(M_2 + 1)$. In the Dirac representation, [7], for the massive spin-$\frac{1}{2}$ fermions, the rotations and the boosts are generated by the matrices

$$D(r_i) = -\frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \qquad D(b_i) = -\frac{1}{2}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}.$$

Analogously, in the fundamental representation $D^{(\frac{1}{2},0)}(G)$ (the other one, $D^{(0,\frac{1}{2})}(G)$, is obtained by the complex conjugation), $D^{(\frac{1}{2},0)}(r_i) = -\frac{1}{2}\mathrm{i}\sigma_i$, $D^{(\frac{1}{2},0)}(b_i) = \frac{1}{2}\sigma_i$. The operator $K(\Gamma = D \otimes D^{(\frac{1}{2},0)\prime})$ and the projector onto its kernel, $G(\Gamma)$, are

$$K(\Gamma) = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \qquad G(\Gamma) = \begin{pmatrix} c & -c \\ -c & c \end{pmatrix}$$

where

$$a = \frac{1}{2}\begin{pmatrix} 5 & 0 & 0 & -2 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ -2 & 0 & 0 & 5 \end{pmatrix} \qquad b = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

$$c = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, the partial trace of $G(\Gamma)$ is the projector

$$G^{(\frac{1}{2},0)}(D) = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Its range, $\mathcal{H}^{(\frac{1}{2},0)}$, is spanned by the vectors satisfying both of the equations $K_1(D)|x\rangle = M_1(M_1 + 1)|x\rangle$ and $K_2(D)|x\rangle = M_2(M_2 + 1)|x\rangle$, with $M_1 = \frac{1}{2}$ and $M_2 = 0$ (or one of them together with (16)). Obviously, the range of $G(\Gamma)$ is spanned by the vector $|(\frac{1}{2}, 0)\,1\rangle = \frac{1}{2}(1, 0, 0, 1, -1, 0, 0, -1)$, giving as the standard subbasis the vectors (the absolute basis takes the role of the vectors $\{|\mu m\rangle\}$)

$$\left\{\left|\left(\frac{1}{2}, 0\right)\,1\,1\right\rangle = \frac{1}{\sqrt{2}}(1, 0, -1, 0), \left|\left(\frac{1}{2}, 0\right)\,1\,2\right\rangle = \frac{1}{\sqrt{2}}(0, 1, 0, -1)\right\}.$$

Analogously, the standard subbasis for $M_1 = 0$ and

$M_2 = \frac{1}{2}$ is $\left\{\left|\left(0, \frac{1}{2}\right)\,1\,1\right\rangle = \frac{1}{\sqrt{2}}(0, -1, 0, -1), \left|\left(0, \frac{1}{2}\right)\,1\,2\right\rangle = \frac{1}{\sqrt{2}}(1, 0, 1, 0)\right\}$.

The last example is the group SU(3). Its eight-dimensional complexified algebra is of rank 2; one Cartan–Weyl's basis is $\{H_1, H_2, E_{\pm \boldsymbol{a}}, E_{\pm \boldsymbol{b}}, E_{\pm \boldsymbol{c}}\}$, where the simple roots are

$$\boldsymbol{b} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \qquad \text{and } \boldsymbol{c} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

while the remaining positive root is $\boldsymbol{a} = (1,0)$. Due to the relation $[E_{\boldsymbol{b}}, E_{\boldsymbol{c}}] = E_{\boldsymbol{a}}$, the representation is given only by the matrices of H_i and simple roots. The Clebsch–Gordan decomposition $D^{(1,0)} \otimes D^{(1,0)} = D^{(0,1)} \oplus D^{(2,0)}$ is considered. For the fundamental representation $D^{(1,0)}$, the matrices are (here E_{ij} is the matrix with elements $(E_{ij})_{pq} = \delta_{ip}\delta_{jq}$)

$$
\begin{aligned}
D^{(1,0)}(H_1) &= \operatorname{diag}\left(\frac{1}{2}, 0, -\frac{1}{2}\right)\\
D^{(1,0)}(H_2) &= \operatorname{diag}\left(\frac{1}{2\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}}\right)\\
D^{(1,0)}(E_{\boldsymbol{b}}) &= \frac{1}{\sqrt{2}} E_{12}\\
D^{(1,0)}(E_{\boldsymbol{c}}) &= \frac{1}{\sqrt{2}} E_{21}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
D^{(2,0)}(H_1) &= \operatorname{diag}\left(1, \frac{1}{2}, 0, 0, -\frac{1}{2}, -1\right)\\
D^{(2,0)}(H_2) &= \operatorname{diag}\left(\frac{1}{\sqrt{3}}, \frac{-1}{2\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}, \frac{-1}{2\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\\
D^{(2,0)}(E_{\boldsymbol{b}}) &= E_{12} + E_{24} + \frac{1}{\sqrt{2}} E_{35}\\
D^{(2,0)}(E_{\boldsymbol{c}}) &= \frac{1}{\sqrt{2}} E_{23} + E_{45} + E_{56}.
\end{aligned}
$$

The second fundamental representation of SU(3) is conjugated to the first one, giving $D^{(0,1)}(l) = -D^{(1,0)^T}(l)$.

The kernel of $K(D^{(1,0)} \otimes D^{(1,0)} \otimes D^{(0,1)'})$ is spanned by the single vector

$$(0, 0, 0, -1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, -1, 0, -1, 0, 0, 0, 1, 0, 0, 0).$$

Also, one vector is found for $K(D^{(1,0)} \otimes D^{(1,0)} \otimes D^{(1,0)})$. These vectors give a standard basis of decomposition:

$$
\begin{aligned}
\Big\{ &\frac{1}{\sqrt{2}}(0, -1, 0, 1, 0, 0, 0, 0, 0), \frac{1}{\sqrt{2}}(0, 0, 1, 0, 0, 0, -1, 0, 0), \frac{1}{\sqrt{2}}(0, 0, 0, 0, 0, -1, 0, 1, 0),\\
&(1, 0, 0, 0, 0, 0, 0, 0, 0), \frac{1}{\sqrt{2}}(0, 1, 0, 1, 0, 0, 0, 0, 0),\\
&\frac{1}{\sqrt{2}}(0, 0, 1, 0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0, 0, 0, 0),\\
&\frac{1}{\sqrt{2}}(0, 0, 0, 0, 0, 1, 0, 1, 0), (0, 0, 0, 0, 0, 0, 0, 0, 1)\Big\}.
\end{aligned}
\tag{21}
$$

As for the Casimir operators, the matrix functions $C^{(\mu)}(D,t)$ or $B^{(\mu)}(D,t)$ are easily found. Expanding them, the matrices $C_2^{(\mu)}(D)$ and $C_3^{(\mu)}(D)$ are found. For example, the eigenvalues of $C_2^{(0,1)}(D)$ are $\frac{2}{3}$ (three times) and $\frac{5}{3}$ (six times), and for $C_3^{(0,1)}(D)$ are $-\frac{7}{9}$ (three times) and $-\frac{5}{18}$ (6 times), which are the values of the operators $C_2^{(0,1)}(D^{(0,1)})$, $C_2^{(0,1)}(D^{(2,0)})$, $C_3^{(0,1)}(D^{(0,1)})$, and $C_3^{(0,1)}(D^{(2,0)})$, respectively. Indeed, the calculation of the generating functions (20) gives (for D irreducible, the scalar matrices must be obtained):

$$B^{(0,1)}(D^{(0,1)},t)=2\frac{1}{1-\frac{1}{3}t}+\frac{1}{1+\frac{2}{3}t}=\sum_{s=0}^{\infty}\left(2\left(\frac{1}{3}\right)^{s}+\left(-\frac{2}{3}\right)^{s}\right)t^{s}$$

$$B^{(0,1)}(D^{(2,0)},t)=\frac{5}{3}\frac{1}{1-\frac{2}{3}t}+\frac{4}{3}\frac{1}{1+\frac{5}{6}t}=\sum_{s=0}^{\infty}\left(\frac{5}{3}\left(\frac{2}{3}\right)^{s}+\frac{4}{3}\left(-\frac{5}{6}\right)^{s}\right)t^{s}.$$

The coefficients with t^2 and t^3 coincide with the mentioned eigenvalues. Of course, the corresponding eigenspaces are the irreducible subspaces. Note that from the expansion of

$$B^{(0,1)}(D^{(1,0)},t)=\frac{8}{3}\frac{1}{1-\frac{1}{6}t}+\frac{1}{3}\frac{1}{1+\frac{4}{3}t}$$

it follows that $C_2^{(0,1)}(D^{(1,0)})=C_2^{(0,1)}(D^{(0,1)})$, and therefore, the operators $C_2^{(\mu)}$ cannot distinguish between these representations (manifesting that the rank of su(3) is 2).

5. Concluding remarks

The modified group projector technique for decomposable representations of the Lie groups is developed in full analogy to the method established for the finite groups. The main object, the subspace $\mathcal{F}^{(\mu)}$ of the fixed points of the representation $\Gamma(G)=D(G)\otimes D^{(\mu)'}(G)$, is characterized either as the range of the group projector $G(\Gamma)$, or as the kernel of the single quadratic Casimir operator $K(\Gamma)$, which naturally emerges at the level of the Lie algebra of G. As for the semisimple groups, the usual Casimir operator technique, aimed to determine the subspace $\mathcal{H}^{(\mu)}$ of the multiple irreducible representation $D^{(\mu)}$, is rederived through the expansion of the operator function $G^{(\mu)}(D,t)$, the partial trace of $G(\Gamma,t)$ over the second space. These functions in the limit $t\to-\infty$ give the group projectors $G^{(\mu)}(D)$ on $\mathcal{H}^{(\mu)}$ and $G(\Gamma)$ on $\mathcal{F}^{(\mu)}$. Also, quite a general formula for the generating functions of the Casimir operators is established.

When the group G is a weak product of its subgroups, $G=G_1G_2$, then $G(\Gamma)=G_1(\Gamma)G_2(\Gamma)$, where the subgroup projectors $G_1(\Gamma)$ and $G_2(\Gamma)$ mutually commute [1]. The Lie algebra of G is the sum of the corresponding subalgebras, and (13) is factorized to the terms with their Casimir operators $K_1(\Gamma)$ and $K_2(\Gamma)$. They satisfy $\mathcal{N}(K(\Gamma))=\mathcal{N}(K_1(\Gamma))\cap\mathcal{N}(K_2(\Gamma))$, which means that instead of $K(\Gamma)$ or $K_1(\Gamma)+K_2(\Gamma)$, both $K_1(\Gamma)$ and $K_2(\Gamma)$ can be used; the system $K_i(\Gamma)|x\rangle=0$ ($i=1,2$) should be considered, giving the corresponding subgroup projectors and the subspaces in the space of $D(G)$. This has been performed in the example of the Lorentz group. Together with the known fact that a pair of opposite roots and an element from Cartan's subalgebra form the sl$(2,\mathbb{C})$ algebra, this result can serve as an easy explanation for extensive usage of 'spins' in physics.

The technique offers a criterion of the irreducibility of the representation $D(G)$. Obviously, the representation is irreducible if and only if the range of $G(D\otimes D')$, i.e. $\mathcal{F}=\mathcal{N}(K(D\otimes D'))$, is a one-dimensional subspace spanned by the vector $|x\rangle=\sum_{i=1}^{|D|}|i\rangle\langle i|$ ($\{|i\rangle\}$ is a basis in the space of $D(G)$). More clearly, using the operators from Hom$(\mathcal{H},\mathcal{H})$ instead of the Dirac notation, this means that $Q(|x\rangle)=I$.

Treating the Lie and finite groups in the uniform way, this technique is in some sense complementary to some other results in such a direction [8, 9]. As or the computer implementations of the group theoretical results in physics, [10], this approach is quite suitable, since it involves only the basis of the algebra. For example, the Clebsch–Gordan coefficients are already contained in the vector spanning the subspace $\mathcal{F}^{(\mu)}$. Indeed, if for the finite-dimensional irreducible representations $D^{(\alpha)}(G)$ and $D^{(\beta)}(G)$ their product

$D(G) = D^{(\alpha)}(G) \otimes D^{(\beta)}(G)$ contains the irreducible component $D^{(\mu)}(G)$ only once (the conditions which must be assumed in the formulation of the problem), then (7) reads

$$|\mu\rangle = \frac{1}{|\mu|} \sum_{a=1}^{|\alpha|} \sum_{b=1}^{|\beta|} \sum_{\mu=1}^{|\mu|} C \begin{pmatrix} \alpha\,\beta\,\mu \\ a\,b\,m \end{pmatrix} |\alpha a\rangle \otimes |\beta b\rangle \otimes \langle \mu' m|$$

and the Clebsch–Gordan coefficients $C\binom{\alpha\,\beta\,\mu}{a\,b\,m}$ can be easily calculated as the scalar products of $|\mu\rangle$ with the uncorrelated basis $|\alpha a\rangle \otimes |\beta b\rangle \otimes \langle \mu' m|$. As for the matrix representations, the last basis is the absolute basis, and the Clebsch–Gordan coefficients are essentially already found, being the coordinates of the normed vector in $\mathcal{F}^{(\mu)}$, multiplied by $|\mu|$.

To compare the efficiency of the proposed and the standard procedure, note that the algorithms for solving the systems of N linear equations and the eigenvalue problem of the square N-dimensional matrix require approximately N^{α} steps, with $\alpha = \log_2 7$ for the best ones, [11]. Therefore, for the semisimple Lie group, with the dimension $|L|$ and rank r, the subspace of the irreducible component $D^{(\mu)}(G)$ of the representation $D(G)$ is obtained, according to the proposed algorithm by (12), within $|L|(|D||\mu|)^{\alpha} + \frac{3}{8}(|D||\mu|)^2$ steps. Within the standard prescription, (18), all of the r Casimir operators are derived within $\sum_{k=2}^{r+1}(k-1)|L|^k(|D|^{\alpha} + |\mu|^{\alpha})$ steps; for each of them, the eigenvalue problem requires further $|D|^{\alpha}$ steps. Thus, the proposed procedure is more efficient by the factor $(1 + (|D|/|\mu|)^{\alpha}) \sum_{k=2}^{r+1}(k-1)|L|^{k-1}$.

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